COL703 - Proof exercises for system $\vdash_{\mathscr{G}}$

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Recall the $\vdash_{\mathscr{G}}$ proof system from the notes.

1. Prove the expressions from the $\vdash_{\mathscr{H}}$ worksheet (for PL) in the propositional fragment of $\vdash_{\mathscr{G}}$.
2. Prove the following expressions in system $\vdash_{\mathscr{G}}$ (without using any context). These expressions are classical FO tautologies.
   (a) $\neg\neg\forall x.\ [\varphi \lor \neg\varphi]$
   (b) $\neg\forall x.\ [\varphi] \supset \exists x.\ [\neg\varphi]$
   (c) $\exists x.\ [\neg\varphi] \supset \neg\forall x.\ [\varphi]$
   (d) $\exists x.\ [\varphi(x) \supset \forall y.\ [\varphi(y)]]$
   (e) $\exists x.\ [\exists y.\ [\varphi(y)] \supset \varphi(x)]$
   (f) $(\forall x.\ [\varphi(x)] \supset \psi) \supset \exists x.\ [\varphi(x) \supset \psi]$, where $x \notin \mathsf{fv}(\psi)$
   (g) $\exists x.\ [\exists y.\ [\varphi(x) \supset \psi(y)]] \supset \exists x.\ [\varphi(x) \supset \psi(x)]$
   (h) $\forall x.\ [\varphi \lor \neg\varphi] \supset \exists x.\ [\varphi] \lor \forall x.\ [\neg\varphi]$
   (i) $\forall x.\ [\varphi \lor \neg\varphi] \land \neg\neg\exists x.\ [\varphi] \supset \exists x.\ [\varphi]$
   (j) $\forall x.\ [\psi \lor \varphi(x)] \supset (\psi \lor \forall x.\ [\varphi(x)])$, where $x \notin \mathsf{fv}(\psi)$
   (k) $(\psi \lor \forall x.\ [\varphi(x)]) \supset \forall x.\ [\psi \lor \varphi(x)]$, where $x \notin \mathsf{fv}(\psi)$
   (l) $\forall x.\ [\neg\neg\varphi] \supset \neg\neg\forall x.\ [\varphi]$
3. Rewrite $\neg\alpha$ as $\alpha \supset \bot$ for any α in the following expressions. Prove them in system $\vdash_{\mathscr{G}}$ with $\bot$ (without using any context). Do not use the $\neg$e rule, but you can use the $\bot$e rule instead. These expressions are intuitionistic FO tautologies.
   (a) $\forall x.\ [\varphi] \supset \exists x.\ [\varphi]$
   (b) $\neg\exists x.\ [\varphi] \supset \forall x.\ [\neg\varphi]$
   (c) $\forall x.\ [\neg\varphi] \supset \neg\exists x.\ [\varphi]$
   (d) $\forall x.\ [\varphi \supset \psi] \supset (\forall x.\ [\varphi] \supset \forall x.\ [\psi])$
   (e) $\forall x.\ [\varphi \supset \psi] \supset (\exists x.\ [\varphi] \supset \exists x.\ [\psi])$
   (f) $\forall x.\ [\varphi(x)] \supset \varphi(t)$
   (g) $(\forall x.\ [\varphi \lor \neg\varphi] \land \neg\neg\forall x.\ [\varphi]) \supset \forall x.\ [\varphi]$
   (h) $\exists x.\ [\psi \land \varphi(x)] \supset (\psi \land \exists x.\ [\varphi(x)])$, where $x \notin \mathsf{fv}(\psi)$
   (i) $(\psi \land \exists x.\ [\varphi(x)]) \supset \exists x.\ [\psi \land \varphi(x)]$, where $x \notin \mathsf{fv}(\psi)$
   (j) $\neg\neg\forall x.\ [\varphi] \supset \forall x.\ [\neg\neg\varphi]$