Lecture 6 - The Hilbert system

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Proof rules: For resolve

$$\frac{\ell_{11} \lor \ell_{12} \lor \ldots \lor \ell_{1m} \lor p \qquad\qquad \ell_{21} \lor \ell_{22} \lor \ldots \lor \ell_{2n} \lor \neg p}{\ell_{11} \lor \ell_{12} \lor \ldots \lor \ell_{1m} \lor \ell_{21} \lor \ell_{22} \lor \ldots \lor \ell_{2n}}\ \mathsf{res}$$

- The horizontal line indicates inference
- The name of the inference rule is given next to the line
- Every expression above the line is called a **premise**
- The expression below the line is called the **conclusion**
- "If all the premises hold, then the conclusion holds"
- Each ℓ_{ij} and p a variable; can substitute any literal and any atom
- Cannot change the "shape" of expressions though

Proof rules

- Mimics inference performed by humans at a **syntactic** level
- "If this and this and this, then that"
- No reference to **semantics** – reasoning purely over syntactic shapes
- But rules need to preserve some manner of semantic soundness
- Can lift this to **proof systems**
- Proof system: specified by a set of axioms and a set of proof rules

Proof systems: Desiderata

- **Purely syntactic**
- **Sound**: Everything that can be inferred using the proof system is a logical consequence of the assumptions
- **Finitary**: Every axiom/proof rule expressible in a finitary manner
- **Decidable**: There is an algorithm which can check, given a set of assumptions and a potential conclusion, if there is a proof of the conclusion from these assumptions using the proof system
- **Complete**: Everything that is a logical consequence of the assumptions can be inferred via some proof in the proof system
- Soundness and completeness tie syntax to semantics!

An axiomatic proof system for PL

- Axiomatic proofs were common in Ancient Greece (see Euclid's Elements of Geometry)
- A notion of a "minimal" axiom system
- The first axiomatic proof system for PL was proposed by Gottlob Frege in his 1879 Begriffsschrift
- Used implication and negation as the connectives for six axioms, along with an inference rule for modus ponens and an implicit substitution
- David Hilbert built on works by Jan Łukasiewicz and Alonzo Church to obtain three schematic axioms and a rule for modus ponens
- This system is called the Hilbert System $\mathscr{H}$

Hilbert System for PL

(H1) $\varphi \supset (\psi \supset \varphi)$

(H2) $(\varphi \supset (\psi \supset \chi)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \chi))$

(H3) $(\neg\varphi \supset \neg\psi) \supset ((\neg\varphi \supset \psi) \supset \varphi)$

$$\frac{\varphi \supset \psi \qquad \varphi}{\psi}\ \text{MP}$$

We denote provability in this system with the symbol $\vdash_{\mathscr{H}}$.

Proofs in system $\mathscr{H}$

- What does a proof in this system look like?
- We will write a proof as an “inverted tree” (actually like a real tree!)
- Goal expression at the root; Leaves on top, labelled by axioms
- Everything in between labelled by an inference rule
- How do we **search** for a proof?
- Start with goal expression, try to match against rules and axioms
- If an axiom schema matches, done!
- Otherwise: apply a rule, look for matches for the premises of the rule

Example

Show that $\vdash_{\mathcal{H}} p \supset p$.

(H1) $\varphi \supset (\psi \supset \varphi)$

(H2) $(\varphi \supset (\psi \supset \chi)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \chi))$

(H3) $(\neg\varphi \supset \neg\psi) \supset ((\neg\varphi \supset \psi) \supset \varphi)$

$$\frac{\varphi \supset \psi \qquad \varphi}{\psi}\,\text{MP}$$

$$\frac{\qquad\qquad\qquad\qquad}{p \supset p}$$

Example

Show that $\vdash_{\mathcal{H}} p \supset p$.

(H1) $\varphi \supset (\psi \supset \varphi)$

(H2) $(\varphi \supset (\psi \supset \chi)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \chi))$

(H3) $(\neg\varphi \supset \neg\psi) \supset ((\neg\varphi \supset \psi) \supset \varphi)$

$$\frac{\varphi \supset \psi \qquad \varphi}{\psi}\,\mathrm{MP}$$

$$\frac{(p \supset \psi) \supset (p \supset p) \qquad\qquad p \supset \psi}{p \supset p}\,\mathrm{MP}$$

Example

Show that $\vdash_{\mathcal{H}} p \supset p$.

(H1) $\varphi \supset (\psi \supset \varphi)$

(H2) $(\varphi \supset (\psi \supset \chi)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \chi))$

(H3) $(\neg\varphi \supset \neg\psi) \supset ((\neg\varphi \supset \psi) \supset \varphi)$

$$\frac{\varphi \supset \psi \qquad \varphi}{\psi}\,\text{MP}$$

$$\frac{\dfrac{(p \supset (\psi \supset p)) \supset (p \supset \psi) \supset (p \supset p) \qquad p \supset (\psi \supset p)}{(p \supset \psi) \supset (p \supset p)}\,\text{MP} \qquad p \supset \psi}{p \supset p}\,\text{MP}$$

Example

Show that $\vdash_{\mathcal{H}} p \supset p$.

(H1) $\varphi \supset (\psi \supset \varphi)$

(H2) $(\varphi \supset (\psi \supset \chi)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \chi))$

(H3) $(\neg\varphi \supset \neg\psi) \supset ((\neg\varphi \supset \psi) \supset \varphi)$

$$\frac{\varphi \supset \psi \qquad \varphi}{\psi}\,\mathrm{MP}$$

$$\dfrac{\dfrac{\dfrac{}{(p \supset (\psi \supset p)) \supset (p \supset \psi) \supset (p \supset p)}\,\mathrm{H2} \qquad \dfrac{}{p \supset (\psi \supset p)}\,\mathrm{H1}}{(p \supset \psi) \supset (p \supset p)}\,\mathrm{MP} \qquad \begin{array}{c}\textbf{???}\\ \vdots \\ p \supset \psi\end{array}}{p \supset p}\,\mathrm{MP}$$

Example

Show that $\vdash_{\mathcal{H}} p \supset p$.

(H1) $\varphi \supset (\psi \supset \varphi)$

(H2) $(\varphi \supset (\psi \supset \chi)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \chi))$

(H3) $(\neg\varphi \supset \neg\psi) \supset ((\neg\varphi \supset \psi) \supset \varphi)$

$$\dfrac{\varphi \supset \psi \qquad \varphi}{\psi}\,\text{MP}$$

Let $\psi = (q \supset p)$ for some $q \in AP$.

$$\dfrac{\dfrac{\dfrac{}{(p \supset (\psi \supset p)) \supset (p \supset \psi) \supset (p \supset p)}\,\text{H2} \qquad \dfrac{}{p \supset (\psi \supset p)}\,\text{H1}}{(p \supset \psi) \supset (p \supset p)}\,\text{MP} \qquad \dfrac{}{p \supset \psi}\,\text{H1}}{p \supset p}\,\text{MP}$$

Exercises

Try to prove the following in $\mathscr{H}$.

- $\neg p \supset (p \supset q)$
- $(\neg q \supset \neg p) \supset ((\neg q \supset p) \supset q))$
- $(p \supset q) \supset (p \supset \neg q) \supset \neg p$
- $\neg\neg p \supset p$

Hilbert system: Soundness

- This system should be sound
- **Theorem (Soundness)**: For any PL expression φ, if $\vdash_{\mathscr{H}} \varphi$, then $\models \varphi$
- If an expression is proved in this proof system, it is a tautology
- Show that each axiom is a validity, and that MP preserves validity
- Can do this via truth tables
- **Exercise**: Prove soundness using meta-theoretic reasoning, without appealing to explicit truth tables.

More about provability in $\mathscr{H}$

- Can weaken the notion of provability to include context
- $\Gamma \vdash_{\mathscr{H}} \varphi$ denotes that there is a proof of φ in System $\mathscr{H}$ using the formulas in Γ as assumptions
- Each "node" in the proof tree will be labelled by a **sequent**, of the form $\Delta \vdash \chi$, where $\Delta \cup \{\chi\} \subseteq \mathsf{PL}$.
- If φ is an instance of H1, H2, or H3, then $\vdash_{\mathscr{H}} \varphi$ and $\Gamma \vdash_{\mathscr{H}} \varphi$ for any Γ
- For any $\varphi \in \Gamma$, $\Gamma \vdash_{\mathscr{H}} \varphi$
- The leaves of any proof $\Gamma \vdash_{\mathscr{H}} \varphi$ are labelled either by instances of H1, H2, or H3, or by formulas that belong to Γ
- **Theorem (Monotonicity)**: If $\Gamma \vdash_{\mathscr{H}} \varphi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash_{\mathscr{H}} \varphi$.
- **Proof idea**: Assume a proof tree for $\Gamma \vdash_{\mathscr{H}} \varphi$. Produce one for $\Gamma' \vdash_{\mathscr{H}} \varphi$.

Example

$\Gamma = \{p \supset q, q \supset r\}$

Show that $\Gamma \vdash_{\mathscr{H}} p \supset r$

(H1) $\varphi \supset (\psi \supset \varphi)$

(H2) $(\varphi \supset (\psi \supset \chi)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \chi))$

(H3) $(\neg\varphi \supset \neg\psi) \supset ((\neg\varphi \supset \psi) \supset \varphi)$

$$\frac{\varphi \supset \psi \qquad \varphi}{\psi}\,\mathrm{MP}$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{H2}
\UnaryInfC{$\Gamma \vdash (p \circ (q \circ r)) \circ ((p \circ q) \circ (p \circ r))$}
\AxiomC{}
\UnaryInfC{$\Gamma \vdash q \circ r$}
\AxiomC{}
\RightLabel{H1}
\UnaryInfC{$\Gamma \vdash (q \circ r) \circ (p \circ (q \circ r))$}
\BinaryInfC{$\Gamma \vdash p \circ (q \circ r)$}
\BinaryInfC{$\Gamma \vdash (p \circ q) \circ (p \circ r)$}
\AxiomC{}
\UnaryInfC{$\Gamma \vdash p \circ q$}
\BinaryInfC{$\Gamma \vdash p \circ r$}
\end{prooftree}
$$