Lecture 3 - Propositional Logic

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COL703 - Logic for Computer Science

1 PL syntax

2 PL semantics

Propositional logic: Syntax

- When using a logic, one is bound by the rules of *syntax*
- Only “grammatically-correct” statements are “allowed”
- Start with a (countable) set *AP* of propositional **atoms**
  - “Smallest” statements of interest
  - Can build up bigger statements with these
- Combine atoms from *AP* using **operators** to form bigger propositions: AND ($\wedge$), OR ($\vee$), NOT ($\neg$), IMPLIES ($\supset$)
- Grammar for propositional logic (PL) is as follows

$$\varphi, \psi := p \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \supset \psi \qquad \text{where } p \in AP$$

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- $\wedge$ and $\vee$ are left-associative; read $\varphi \wedge \psi \wedge \chi$ as $(\varphi \wedge \psi) \wedge \chi$
- $\supset$ is right-associative; read $\varphi \supset \psi \supset \chi$ as $\varphi \supset (\psi \supset \chi)$
- This grammar produces the **well-formed formulas** (wffs) of propositional logic
- Can construct abstract syntax trees (ASTs) for well-formed formulas
- Examples:
  - $\neg(\varphi \wedge \psi) \vee (\neg\chi \vee \neg\varphi)$
  - $\neg((\neg\varphi) \wedge \neg(\psi \vee \neg\chi))$
- **Exercise**: Show that each wff in PL has a unique AST

AST for example

Example: $\neg(\varphi \land \psi) \lor (\neg\chi \lor \neg\varphi)$

AST for example

Example: $\neg((\neg\varphi) \land \neg(\psi \lor \neg\chi))$

More about wffs

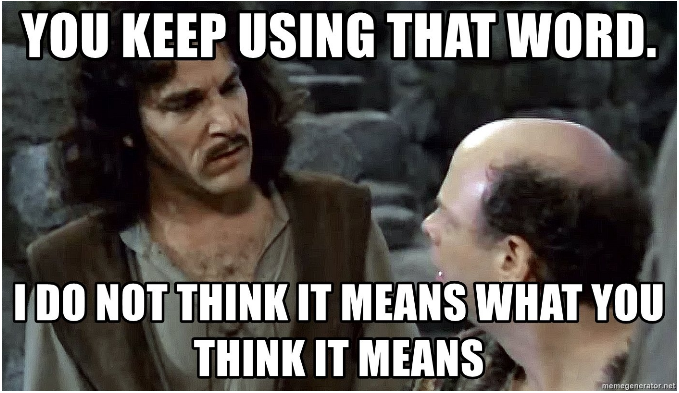
- **Main connective** of a wff: Labels the root of the AST
- Define the **set of subformulae** of a wff φ (denoted $\mathsf{sf}(\varphi)$) as follows
  - $\mathsf{sf}(p) = \{p\}$, for every $p \in AP$
  - $\mathsf{sf}(\neg\varphi) = \{\neg\varphi\} \cup \mathsf{sf}(\varphi)$
  - $\mathsf{sf}(\varphi \circ \psi) = \{\varphi \circ \psi\} \cup \mathsf{sf}(\varphi) \cup \mathsf{sf}(\psi)$, for $\circ \in \{\wedge, \vee, \supset\}$
- Can define the **set of atoms** of a wff inductively as well (More easily?)
- Define the **size** of a wff φ (denoted $\mathsf{size}(\varphi)$) as follows
  - $\mathsf{size}(p) = 1$, for every $p \in AP$
  - $\mathsf{size}(\neg\varphi) = 1 + \mathsf{size}(\varphi)$
  - $\mathsf{size}(\varphi \circ \psi) = 1 + \mathsf{size}(\varphi) + \mathsf{size}(\psi)$, for $\circ \in \{\wedge, \vee, \supset\}$
- We will usually define these notions for wffs in any logic via induction on the structure of formulae, as done here.

Example: Tic-Tac-Toe

- Given:
  - A 3×3 grid of nine squares
  - Each square can have a circle ($\bigcirc$) or a cross ($\times$)
- What does it mean to win?
- What does it mean to have an invalid grid?

What next?

- Now I can churn out the set of all PL wffs; so what?
- Would like to **manipulate symbols** to make sense of the world
- Want it to correspond to the **manipulation of meaning**
- A symbol can mean whatever you choose it to mean; what then?



Assigning meaning to atoms

- We have a countable list of atoms, which are basic facts about our world
- Index each atom by a natural number $p_0, p_1, p_2, \ldots$
- Map each fact about the world to an atom p_i
- As long as our inference is **sound**, we can operate in syntax!
- The actual interpretation of atoms is extraneous to the process
- Interpretations of **atoms can vary**
- Interpretations of **connectives must stay fixed**

Truth and falsehood

- A proposition is a statement that can be evaluated for truth or falsehood.
- Natural to talk about assigning a *truth value* to a proposition
- A proposition has a truth value which is one of “True” or “False”
- Some function $\tau : AP \to \{T, F\}$ assigns truth values to atoms
  - Ideally, τ closely mirrors the “real world”
  - Not necessary, but obviously desirable!
  - Such a τ is called a **valuation**!
- Build up truth values for wffs using these; How?
  - Could use truth tables to look it up for each case
  - Better: use induction to define a construction

Truth values: Inductive definition

- Define the **truth value** of a wff φ (denoted $[\![\varphi]\!]_\tau$) as follows:

$$[\![p]\!]_\tau = \tau(p), \text{ where } \tau(p) \in \{T, F\}$$

$$[\![\neg\varphi]\!]_\tau = \begin{cases} F & \text{if } [\![\varphi]\!]_\tau = T \\ T & \text{if } [\![\varphi]\!]_\tau = F \end{cases}$$

$$[\![\varphi \wedge \psi]\!]_\tau = \begin{cases} T & \text{if } [\![\varphi]\!]_\tau = T \text{ and } [\![\psi]\!]_\tau = T \\ F & \text{otherwise} \end{cases}$$

$$[\![\varphi \vee \psi]\!]_\tau = \begin{cases} T & \text{if } [\![\varphi]\!]_\tau = T \text{ or } [\![\psi]\!]_\tau = T \\ F & \text{otherwise} \end{cases}$$

$$[\![\varphi \supset \psi]\!]_\tau = \begin{cases} F & \text{if } [\![\varphi]\!]_\tau = T \text{ and } [\![\psi]\!]_\tau = F \\ T & \text{otherwise} \end{cases}$$

More about truth values

- We defined a valuation τ as a (total) function from AP to $\{T, F\}$
- Countably many elements in AP
- But a formula is a finite object
- Do we need to carry all this information around in τ?
- **Exercise**: Show that for any formula φ, $[\![\varphi]\!]_\tau = [\![\varphi]\!]_{\tau'}$, where τ' is the partial function obtained by restricting τ to the atoms of φ.

Satisfiability & Validity

- τ is called a **model** for a formula φ (denoted $\tau \models \varphi$) iff $[\![\varphi]\!]_\tau = T$.
- φ is said to be **satisfiable** if it has **at least one** model, and **unsatisfiable** otherwise; can lift this to sets
- For a finite set $\Gamma = \{\varphi_i \mid 0 \leqslant i \leqslant n\}$ of formulae, $\tau \models \Gamma$ iff $\tau \models \bigwedge_{0 \leqslant i \leqslant n} \varphi_i$
- φ is said to be **valid** if **every** valuation τ is a model for φ
- A valid formula is often also called a *tautology*
- Sometimes we use the set ν of atomic propositions assigned true by τ as the model (in which case we also write $\nu \models \varphi$)
- **Exercise**: Write an inductive definition for $\nu \models \varphi$ (closely follow the definition of $[\![\varphi]\!]_\tau$)

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Exercise: Which of the following should be satisfiable/unsatisfiable/valid?

- At least one square has a circle

Example: Tic-Tac-Toe

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- At least one square has a circle
- At least one square has a cross

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- A square cannot simultaneously contain both a circle and a cross

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- The number of circles is less than that of crosses

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- The number of crosses is less than or equal to that of circles

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- The difference between the number of occurrences of the two symbols is at most one
- If three occurrences of ○ line up horizontally, vertically, or diagonally, then the number of occurrences of × is no more than that of ○, and vice versa

More about satisfiability and validity

- These need us to check all valuations/truth values
- To see if at least one/all of them satisfy the formula
- How many different valuations are possible for a given formula?
- Same as the number of rows in the truth table
- This is a (terrible) function of the number of atoms in the formula!
- Can we do something better?

Logical consequence

- What does it mean for a valuation τ to be a model of a formula φ?
- τ makes some atomic propositions true, and also makes φ true
- A proposition φ is called a **logical consequence** of a set Γ of propositions if any valuation that is a model for Γ is also a model for φ
- Slightly overload notation to denote this also by $\Gamma \models \varphi$ (even though Γ can contain non-atomic formulas)
- For an empty Γ, logical consequence is nothing but validity