Lecture 22 - Incompleteness

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Recap

- Wanted to reduce truth to provability in our proof system
- What if I wanted to obtain every fact that is true of $\mathbb{N}$?
- Consider all sentences true of the natural numbers: $\mathtt{Th}(\mathbb{N})$
- Löwenheim-Skolem says: There is also an **uncountable** model which satisfies these sentences
- So $\mathtt{Th}(\mathbb{N})$ is satisfied by multiple models of various cardinalities.

So far...

- FO Completeness: every truth can be proven using $\vdash_{\mathcal{G}}$
- Every truth about groups can be derived using γ_{grps} as the hypothesis
- Can do this for all the γs that we saw
- We are often interested in specific structures
- What about $\mathbb{R}$? $\mathbb{Q}$? $\mathbb{N}$?
- Is there some axiomatization of $\mathbb{N}$ such that one can derive all truths about the naturals from it?
- Can I derive, for example, the following sentence?

$$\forall x.\ [P(x) \land \exists y.\ [x \equiv 2 \times y] \supset P(y)] \land$$
$$\forall x.\ [P(x) \land \exists y.\ [x \equiv 2 \times y + 1] \supset P(3 \times x + 1)] \supset P(1)$$

Some history

- Sunday, the 7th of September, 1930, in a small conference on the foundations of mathematics in Königsberg
- Kurt Gödel presents his completeness result, from his PhD work
- Casually follows it up with a rather abstruse statement about consistency and provability of false statements.

> *One can (under the assumption of the consistency of classical mathematics) even give examples of statements (and even such of the sort of Goldbach's or Fermat's) which are conceptually correct but unprovable in the formal system of classical mathematics. Therefore, if one adjoins the negation of such a statement to the axioms of classical mathematics, then one obtains a consistent system in which a conceptually false statement is provable.*

A semi-formal statement of incompleteness

Suppose S is an effectively axiomatized formal theory whose language contains the language of basic arithmetic. Then, if S is consistent, and can prove a certain amount of arithmetic, there will be a sentence κ of basic arithmetic such that $\mathbb{N} \vDash \kappa$ and $S \nvdash \kappa$

Context: Hilbert's programme

- Two formal theories of mathematics, S and T
- S: finite, meaningful statements, and "nice" methods of proof
- T: transfinite, idealized statements and methods
- Goal: Show that for any φ, if $T \vdash \varphi$, then $S \vdash \varphi$

Context: Hilbert's programme

- Two formal theories of mathematics, S and T
- S: finite, meaningful statements, and "nice" methods of proof
- T: transfinite, idealized statements and methods
- Goal: Show that for any φ, if $T \vdash \varphi$, then $S \vdash \varphi$ using the methods in S
- Gödel showed that there is a true φ (so $T \vdash \varphi$) but $S \nvdash \varphi$ for any "nice" S: **First Incompleteness**
- **Second Incompleteness** takes this a step further: There is a particular φ (namely, that S is consistent) which cannot be proved in S

Peano axioms PA

$$\Sigma = (\{0\}, \{s/1, +/2, \times/2\}, \emptyset)$$

$$
\begin{array}{ll}
(A1) & \forall x.\ [\neg(0 \equiv s(x))] \\
(A2) & \forall x.\ [\forall y.\ [(s(x) \equiv s(y)) \supset (x \equiv y)]] \\
(A3) & \forall x.\ [x + 0 \equiv x] \\
(A4) & \forall x.\ [\forall y.\ [x + s(y) \equiv s(x + y)]] \\
(A5) & \forall x.\ [x \times 0 \equiv 0] \\
(A6) & \forall x.\ [\forall y.\ [(x \times s(y)) \equiv x + (x \times y)]] \\
(A7_\varphi) & \varphi(0) \supset \forall x.\ [\varphi(x) \supset \varphi(s(x))] \supset \forall x.\ [\varphi(x)]
\end{array}
$$

- Infinite; one $(A7_\varphi)$ for every formula $\varphi \in \mathsf{FO}_\Sigma$ with one free variable
- We say $\mathsf{PA} \vdash \alpha$ iff there is a proof of α using the above system and $\vdash_{HK}$
- $\mathsf{PA} \vdash \alpha$ implies $\mathbb{N} \vDash \alpha$

Proof sketch

- **Incompleteness Theorem (Gödel, 1931)**: No recursive, consistent extension of PA is complete.
- No "nice" axiom system is adequate to prove all truths about $\mathbb{N}$
- Gödel's original idea: Provability in PA is programmable! Truth is not.
- So $\{\varphi \mid PA \vdash \varphi\} \neq \{\varphi \mid \mathbb{N} \models \varphi\}$
- Gödel showed the former is definable by an expression, not the latter
- How can an expression in arithmetic define a set of expressions?
- Arithmetization: Code all formulas as numbers.
- Any expression defines some property over numbers, so we good!

Proof details

- $n \in \mathbb{N}$ represented by $\overline{\mathsf{n}}$ in $\mathrm{T}(\Sigma)$ (nothing but s applied n times to $\mathsf{0}$)
- Everything in the logical language appears in sans-serif blue
- The proof system PA and natural numbers appear in *brown*
- What does it mean for an expression to be definable in this language?
- When is a k-ary relation $R \subseteq \mathbb{N}^k$ over the naturals definable?
- Iff there is a formula φ_R with k free variables such that for all $n_1, n_2, \ldots, n_k \in \mathbb{N}$, we have $(n_1, n_2, \ldots, n_k) \in R$ iff $\mathbb{N} \vDash \varphi_R\left(\overline{\mathsf{n}_1}, \overline{\mathsf{n}_2}, \ldots, \overline{\mathsf{n}_k}\right)$
- Similarly, a function $f : \mathbb{N}^k \to \mathbb{N}$ is definable iff there is a formula φ_f with $k+1$ free variables such that for all $n_1, n_2, \ldots, n_k, m \in \mathbb{N}$, we have $f(n_1, n_2, \ldots, n_k) = m$ iff $\mathbb{N} \vDash \varphi_f\left(\overline{\mathsf{n}_1}, \overline{\mathsf{n}_2}, \ldots, \overline{\mathsf{n}_k}, \overline{\mathsf{m}}\right)$

Proof details: Arithmetization

- A **Gödel-numbering scheme** is some effective way of coding up expressions in PA (and sequences thereof) as natural numbers
- Given a Gödel-numbering scheme, the code for an expression (or a sequence thereof) is its unique **Gödel number** (in **bold purple**)
- There is a Gödel-numbering scheme for PA
- Can decide:
  - whether an expression is well-formed and whether it is a sentence
  - whether a given $\mathbf{n}$ codes up a well-formed expression or a sentence
- We denote by $\delta_{\mathbf{n}}$ the expression coded up by $\mathbf{n}$

Proof details: Gödel numbering

- How exactly does one assign Gödel numbers?
- Arbitrary coding for basic building blocks (variables and symbols in Σ)
- Extend to sequences of symbols/terms/expressions using exponentiation and primes, using the following lemma
- **Gödel's β-function lemma**: There is a PA-definable function $\beta : \mathbb{N}^3 \to \mathbb{N}$ s.t. for every $n \geqslant 0$ and every sequence $a_0 \dots a_{n-1}$, there are $c, d \in \mathbb{N}$ s.t. for all $i < n$, $a_i = \beta(c, d, i)$.
- One can then define the following predicates:
  - $\mathsf{Seq}\left(\overline{\mathsf{m}}\right)$: **m** codes a sequence of numbers
  - $\mathsf{Len}\left(\overline{\mathsf{m}}, \overline{\mathsf{n}}\right)$: **m** codes a sequence of length n
  - $\mathsf{Elem}\left(\overline{\mathsf{m}}, \overline{\mathsf{i}}, \overline{\mathsf{n}}\right)$: **m** codes a sequence whose i^{th} element is n

About proof in PA

- There is a wff $\text{Proof}(x, y)$ in the language of basic arithmetic such that $\text{Proof}(\mathbf{m}, \mathbf{n})$ is true iff $\mathbf{m}$ codes up a PA-proof of $\delta_{\mathbf{n}}$
- What is a proof in PA?

About proof in PA

- There is a wff $\mathsf{Proof}(x, y)$ in the language of basic arithmetic such that $\mathsf{Proof}(\mathbf{m}, \mathbf{n})$ is true iff $\mathbf{m}$ codes up a PA-proof of $\delta_{\mathbf{n}}$
- What is a proof in PA? A sequence of expressions such that each expression is either an axiom (either of FO or of PA) or follows from some earlier expression(s) using a proof rule.
- Each expression in this sequence has its own Gödel number
- Different elements of sequence are related to each other using Elem
- Predicate $\mathsf{ValidProof}(x)$ says that x is a sequence (via Seq) and captures the above two statements.
- Predicate to say that x is a proof of y:

$$\mathsf{Proof}(x, y) \coloneqq \mathsf{ValidProof}(x) \wedge \exists k.\ [\mathsf{Len}(x, k) \wedge \mathsf{Elem}\,(x, k, y)]$$

Provability $\neq$ truth

- Provability: $\mathsf{Prov}(y) \coloneqq \exists x.\ [\mathsf{Proof}(x, y)]$. $\mathbb{N} \vDash \mathsf{Prov}(m)$ iff $\mathsf{PA} \vdash \delta_{\mathbf{m}}$
- We will now show that there is no corresponding truth predicate $\mathsf{True}(x)$ s.t. $\mathbb{N} \vDash \mathsf{True}(\overline{\mathsf{m}})$ iff $\mathbb{N} \vDash \delta_{\mathbf{m}}$
- Define $\mathsf{Diag}\,(x, y)$ s.t. $\mathbb{N} \vDash \mathsf{Diag}\left(\overline{\mathsf{m}}, \overline{\mathsf{p}}\right)$ iff $\delta_{\mathbf{m}}\left\{\overline{\mathsf{m}}/v_0\right\} = \delta_{\mathbf{p}}$ (where v_0 is the first variable in our enumeration of variables)
- Suppose there is a truth predicate $\mathsf{True}(x)$
- Then, we can define $\psi(v_0) \coloneqq \exists x.\ [\mathsf{Diag}\,(v_0, x) \wedge \neg\mathsf{True}(x)]$
- Let d be such that $\psi = \delta_{\mathbf{d}}$. Let $\kappa \coloneqq \psi\left(\overline{\mathsf{d}}\right)$, and let h be such that $\kappa = \delta_{\mathbf{h}}$.
- **Exercise**: Prove that $\mathbb{N} \vDash \forall y.\ \left[\mathsf{Diag}\left(\overline{\mathsf{d}}, y\right) \leftrightarrow \left(y \equiv \overline{\mathsf{h}}\right)\right]$
- Now, apply a usual diagonalization argument, to get a contradiction.

Provability ≠ truth: Diagonalization

$$
\begin{aligned}
& \mathbb{N} \models \kappa \\
\text{iff} \quad & \mathbb{N} \models \psi\left(\overline{\mathsf{d}}\right) \\
\text{iff} \quad & \mathbb{N} \models \exists x. \left[\mathsf{Diag}\left(\overline{\mathsf{d}}, x\right) \wedge \neg \mathsf{True}(x)\right] \\
\text{iff} \quad & \mathbb{N} \models \neg \mathsf{True}\left(\overline{\mathsf{h}}\right) \quad \textbf{Exercise}: \text{Verify this iff} \\
\text{iff} \quad & \mathbb{N} \not\models \mathsf{True}\left(\overline{\mathsf{h}}\right) \\
\text{iff} \quad & \mathbb{N} \not\models \delta_{\mathbf{h}} \\
\text{iff} \quad & \mathbb{N} \not\models \kappa
\end{aligned}
$$

About the choice of system

- There are more truths than provable expressions
- These truths are not "unprovable at all"; just **unprovable in** PA
- What if we add some of these truths as extra axioms into PA?
- Suppose we get PA' by doing this
- PA' is still "nice", because provability in PA' is still definable in arithmetic
- So repeat the same argument, and show that PA' is also incomplete!
- Less an incomplete**ness** theorem, more an incomplet**ability** theorem