Lecture 21 - More About First-Order Theories

Vaishnavi Sundararajan

COL703 - Logic for Computer Science

Recap

- Looked at a few theories of "common" constructions
- Groups, fields, orders...
- Saw that one can define formulae which characterize subclasses of these constructions (groups with no elements of order 2 etc)
- Captured the $<$ relation as a formula with two free variables in $(\mathbb{R}, +, \times, 0)$
- Saw that $<$ **cannot** be captured in $(\mathbb{R}, +, 0)$
- Used an automorphism to show this

Elementary classes

- We showed that $<$ is not definable using the signature without $\times$
- But we also defined entire classes of groups, fields etc via FO formulas
- How to show if an entire class of models is characterizable using FO?
- For a set X of Σ-sentences, we define

$$\mathrm{Mod}\, X := \left\{ \mathcal{M} \;\middle|\; \mathcal{M} \text{ is a } \Sigma\text{-structure, and } \mathcal{M} \models X \right\}$$

- Let $\mathscr{C}$ be a class of Σ-structures. $\mathscr{C}$ is said to be
  - **elementary** if there is a $\varphi \in \mathrm{FO}_\Sigma$ such that $\mathscr{C} = \mathrm{Mod}\,\{\varphi\}$
  - **Δ-elementary** if there is a set $X \subseteq \mathrm{FO}_\Sigma$ such that $\mathscr{C} = \mathrm{Mod}\, X$.
- **Elementary**: An FO sentence φ captures the exact class of models

Examples

- The class $\mathscr{C}_{\text{grps}}$ of groups is elementary.

Examples

- The class $\mathscr{C}_{\text{grps}}$ of groups is elementary. $\mathscr{C}_{\text{grp}} = \text{Mod}\ \gamma_{\text{grps}}$
- Classes of equivalence relations, orders, and fields also elementary
- Let p be a prime. A field F has **characteristic** p if $\underbrace{1 + \cdots + 1}_{p \text{ times}} = 0$. If there is no such p, then F has characteristic 0.
- The field $\mathbb{R}$ of real numbers has characteristic 0.
- Let $\chi_p \coloneqq \underbrace{1 + \cdots + 1}_{p \text{ times}} \equiv 0$
- The class of fields of characteristic p is $\text{Mod}\ (\gamma_{\text{flds}} \wedge \chi_p)$
- The class $\mathscr{C}$ of fields of characteristic 0 is Δ-elementary
- $\mathscr{C} = \text{Mod}\ \{\gamma_{\text{flds}}\} \cup \{\neg\chi_p \mid p \text{ is a prime}\}$
- Is $\mathscr{C}$ elementary?

Examples

- The class $\mathscr{C}_{\text{grps}}$ of groups is elementary. $\mathscr{C}_{\text{grp}} = \text{Mod}\ \gamma_{\text{grps}}$
- Classes of equivalence relations, orders, and fields also elementary
- Let p be a prime. A field F has **characteristic** p if $\underbrace{1 + \cdots + 1}_{p \text{ times}} = 0$. If there is no such p, then F has characteristic 0.
- The field $\mathbb{R}$ of real numbers has characteristic 0.
- Let $\chi_p \coloneqq \underbrace{1 + \cdots + 1}_{p \text{ times}} \equiv 0$
- The class of fields of characteristic p is $\text{Mod}\ (\gamma_{\text{flds}} \wedge \chi_p)$
- The class $\mathscr{C}$ of fields of characteristic 0 is Δ-elementary
- $\mathscr{C} = \text{Mod}\ \{\gamma_{\text{flds}}\} \cup \{\neg\chi_p \mid p \text{ is a prime}\}$
- Is $\mathscr{C}$ elementary? We can use Compactness to show that it is **not**.

Compactness theorem

- **Compactness Theorem**: A set Γ of FO sentences is satisfiable iff every finite subset of Γ is satisfiable.
- **Proof**: Suppose Γ satisfiable. Then all finite subsets of Γ also satisfiable.
- Now suppose that Γ is not satisfiable. We know that every consistent set is satisfiable. So Γ is not consistent.
- So there is some $\{\varphi_1, \dots, \varphi_n\} \subseteq_{\text{fin}} \Gamma$ such that $\vdash \neg(\varphi_1 \wedge \dots \wedge \varphi_n)$
- But by Soundness, $\models \neg(\varphi_1 \wedge \cdots \wedge \varphi_n)$
- So there is a finite subset of Γ that is unsatisfiable.

Compactness: Application

- Let φ be a sentence which holds in all fields of characteristic 0
- So $\{\gamma_{\text{flds}}\} \cup \{\neg\chi_p \mid p \text{ is a prime}\} \models \varphi$
- Compactness tells us that there is some n_0 such that $\{\gamma_{\text{flds}}\} \cup \{\neg\chi_p \mid p \text{ is a prime}, p < n_0\} \models \varphi$
- Hence, φ is valid in all fields of characteristic $\geqslant n_0$!
- So, a sentence which is valid in all fields of characteristic 0 is also valid in all fields with a "sufficiently large" characteristic.
- So the class of fields with characteristic 0 is **not** elementary

Elementary equivalence

- So far, we saw some classes of structures that FOL can characterize
- What about which classes can be distinguished via FOL sentences?
- When are two structures **not** distinguishable?
- When they satisfy the same sentences
- Two Σ-structures $\mathscr{M}$ and $\mathscr{M}'$ are said to be **elementarily-equivalent** (denoted $\mathscr{M} \bowtie \mathscr{M}'$) if for every sentence $\varphi \in \mathsf{FO}_\Sigma$, $\mathscr{M} \models \varphi$ iff $\mathscr{M}' \models \varphi$
- For a Σ-structure $\mathscr{M}$, the **theory** of $\mathscr{M}$ is the set of sentences it satisfies: $\mathrm{Th}(\mathscr{M}) = \{\varphi \mid \mathscr{M} \models \varphi\}$.
- **Theorem**: For two Σ-structures $\mathscr{M}$ and $\mathscr{M}'$, $\mathscr{M} \bowtie \mathscr{M}'$ iff $\mathscr{M}' \models \mathrm{Th}(\mathscr{M})$.
- **Exercise**: Prove this statement.

Elementary equivalence

- Clear that any two isomorphic structures are elementarily-equivalent.
- Are any two elementarily-equivalent structures also isomorphic?
- Can we say something about the class of structures that are all elementarily-equivalent to a particular $\mathscr{M}$?
- **Theorem**: For any $\mathscr{M}$, $\mathscr{C} = \{\mathscr{M}' \mid \mathscr{M} \bowtie \mathscr{M}'\} = \text{Mod Th}(\mathscr{M})$ is Δ-elementary. $\mathscr{C}$ is the smallest Δ-elementary class which contains $\mathscr{M}$.
- **Exercise**: Prove this!
- Is the class of all structures isomorphic to $\mathscr{M}$ also Δ-elementary?
- Does the cardinality of $\mathscr{M}$ influence the answer?
- Suppose I have an uncountable $\mathscr{M}$ (over a countable Σ)
- What can I say about all structures elementarily-equivalent to $\mathscr{M}$?

Downward Löwenheim-Skolem theorem

- **(Downward) Löwenheim-Skolem Theorem**: If a set Γ of sentences over a countable Σ is satisfiable, it is satisfied by a countable model
- **Proof**: Consider a satisfiable set Γ of FO sentences over a countable Σ.
- Γ is consistent. **Exercise**: Prove this!
- For Completeness, we built a model whose elements were equivalence classes of terms of the language
- What is the cardinality of $T(\Sigma)$ for a countable Σ?

Downward Löwenheim-Skolem theorem

- **(Downward) Löwenheim-Skolem Theorem**: If a set Γ of sentences over a countable Σ is satisfiable, it is satisfied by a countable model
- **Proof**: Consider a satisfiable set Γ of FO sentences over a countable Σ.
- Γ is consistent. **Exercise**: Prove this!
- For Completeness, we built a model whose elements were equivalence classes of terms of the language
- What is the cardinality of $T(\Sigma)$ for a countable Σ?
- How many equivalence classes can there be over a countable set?

Downward Löwenheim-Skolem theorem

- **(Downward) Löwenheim-Skolem Theorem**: If a set Γ of sentences over a countable Σ is satisfiable, it is satisfied by a countable model
- **Proof**: Consider a satisfiable set Γ of FO sentences over a countable Σ.
- Γ is consistent. **Exercise**: Prove this!
- For Completeness, we built a model whose elements were equivalence classes of terms of the language
- What is the cardinality of $T(\Sigma)$ for a countable Σ?
- How many equivalence classes can there be over a countable set?
- At most countably many
- Thus, every satisfiable set of sentences (over a countable signature) is consistent, and satisfiable in a countable model!

Downward L-S: Application

Theorem: Let Σ be countable, and $X \in \mathsf{FO}_\Sigma$ be a set of sentences which has arbitrarily large finite models (i.e. for every $n \in \mathbb{N}$ there is a model for X with cardinality at least n). Then, X is satisfied by a countably infinite model.

Proof: Recall $\varphi_{\geqslant n} = \exists x_1.\left[\exists x_2.\left[\ldots \exists x_n.\left[\bigwedge_{1\leqslant i<j\leqslant n} \neg(x_i \equiv x_j)\right]\ldots\right]\right]$.

$$\text{Define} \quad Y \coloneqq X \cup \{\varphi_{\geqslant m} \mid m \geqslant 2\}$$

Every model of Y is infinite, and also a model of X.

Exercise: Is there even one such?

So Y (and consequently X) is satisfied by a countably infinite model, by downward L-S.

Upward Löwenheim-Skolem theorem

- **(Upward) Löwenheim-Skolem Theorem**: If a set $\Gamma \subseteq \mathsf{FO}_\Sigma$ is satisfied by an infinite model, then, for any set A, there is a model for Γ which has at least as many elements as A.
- **Proof**: For each $a \in A$, let $c_a \notin \mathscr{C}$ be a new constant such that $c_a \neq c_b$ for distinct $a, b \in A$. Let $\Sigma' = (\mathscr{C} \cup \{c_a \mid a \in A\}, \mathscr{F}, \mathscr{P})$.
- Let $G := \Gamma \cup \{\neg(c_a \equiv c_b) \mid a, b \in A, a \neq b\} \subseteq \mathsf{FO}_{\Sigma'}$. Suppose $\mathscr{I} \models G$.
- $\mathscr{I}$ is also a model for Γ
- Clear that $\mathscr{I}(c_a) \neq \mathscr{I}(c_b)$ (since $\mathscr{I} \models \neg(c_a \equiv c_b)$ for distinct $a, b \in A$)
- Then, $\{(a, \mathscr{I}(a)) \mid a \in A\}$ is an injective map from A to the domain of $\mathscr{I}$, and so the model $\mathscr{I}$ for G has at least as many elements as A.
- **Exercise**: Show that G is satisfiable.

Löwenheim-Skolem theorem

- By Downward L-S, an **uncountable** $\mathscr{M}$ has an elementarily-equivalent **countable** $\mathscr{M}'$
- $\mathscr{M}$ and $\mathscr{M}'$ are clearly not isomorphic
- So elementary equivalence and isomorphism do not coincide
- The class of all structures isomorphic to $\mathscr{M}$ is **not** Δ-elementary
- By Upward L-S, a **countable** $\mathscr{M}$ has an elementarily-equivalent **uncountable** $\mathscr{M}'$

So why did we do all this?

- Recall that we wanted to reduce truth to provability in our proof system
- Common enough setting: natural numbers
- What if I wanted to obtain every fact that is true about $\mathbb{N}$?
- Consider all sentences true of the natural numbers: $\mathtt{Th}(\mathbb{N})$
- There is also an **uncountable** model which satisfies these sentences!
- So $\mathtt{Th}(\mathbb{N})$ is satisfied by multiple models of various cardinalities.
- But do we need $\mathtt{Th}(\mathbb{N})$ to have exactly one model?

So why did we do all this?

- Recall that we wanted to reduce truth to provability in our proof system
- Common enough setting: natural numbers
- What if I wanted to obtain every fact that is true about $\mathbb{N}$?
- Consider all sentences true of the natural numbers: $\mathrm{Th}(\mathbb{N})$
- There is also an **uncountable** model which satisfies these sentences!
- So $\mathrm{Th}(\mathbb{N})$ is satisfied by multiple models of various cardinalities.
- But do we need $\mathrm{Th}(\mathbb{N})$ to have exactly one model? Not necessarily
- All we want is a “nice” set of axioms $\Gamma_{\mathbb{N}}$ such that $\varphi \in \mathrm{Th}(\mathbb{N})$ iff $\Gamma_{\mathbb{N}} \vdash_{\mathscr{G}} \varphi$

So why did we do all this?

- Recall that we wanted to reduce truth to provability in our proof system
- Common enough setting: natural numbers
- What if I wanted to obtain every fact that is true about $\mathbb{N}$?
- Consider all sentences true of the natural numbers: $\mathrm{Th}(\mathbb{N})$
- There is also an **uncountable** model which satisfies these sentences!
- So $\mathrm{Th}(\mathbb{N})$ is satisfied by multiple models of various cardinalities.
- But do we need $\mathrm{Th}(\mathbb{N})$ to have exactly one model? Not necessarily
- All we want is a "nice" set of axioms $\Gamma_{\mathbb{N}}$ such that $\varphi \in \mathrm{Th}(\mathbb{N})$ iff $\Gamma_{\mathbb{N}} \vdash_{\mathscr{G}} \varphi$
- But Gödel's incompleteness theorem says that **no such $\Gamma_{\mathbb{N}}$ exists**.

Quiz