Lecture 18 - Natural Deduction

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Recap: Completeness of \vdash_{HK}

- Gödel's Completeness Theorem (1929): If $\Gamma \models \varphi$, then $\Gamma \vdash_{HK} \varphi$
- Equivalent statement: Any consistent set of expressions is satisfiable
- Lindenbaum's Lemma: Every consistent set can be extended to an -fulfilled maximally consistent set (MCS).
- Show a model for an \exists -fulfilled MCS.
- So every consistent set can be extended to an ∃-fulfilled MCS which is satisfiable.
- Same model satisfies the original consistent set (contained) also.

Proof system

- We have shown that \vdash_{HK} is a complete proof system for FOL.
- It is not a particularly intuitive proof system though.
- **Exercise**: Try to prove $\vdash_{HK} \exists x. [x \equiv x]$
- Everything has to be cast in terms of ¬, ⊃, and ∀
- One needs to know which instances of which axioms to use
- Would like a proof system that more closely mirrors human reasoning
- Fewer axioms, more proof rules!
- Gerhard Gentzen introduced one such, called "Natural Deduction"

Natural Deduction

- No axiom schema, only proof rules
- No need to worry about which instances of which axioms
- Each operator gets an introduction rule and/or an elimination rule
- Introduction rule: Operator appears in the conclusion
- Elimination rule: Operator appears in the (RHS of) premise(s), does not appear in the conclusion

Proof rules for propositional fragment

Introduction rule	Elimination rule
$\frac{\Gamma \vdash \varphi_0 \qquad \Gamma \vdash \varphi_1}{} \land i$	$\frac{\Gamma \vdash \varphi_0 \land \varphi_1}{\Box} \land e_1$
$\Gamma \vdash \varphi_0 \land \varphi_1$	$\frac{\Gamma \vdash \varphi_j}{\Gamma \vdash \varphi_j} \wedge e_j$
$\frac{\Gamma \vdash \varphi_j}{\bigvee i_i}$	$\frac{\Gamma \vdash \varphi_0 \lor \varphi_1 \qquad \Gamma, \varphi_0 \vdash \psi \qquad \Gamma, \varphi_1 \vdash \psi}{} \lor e$
$\Gamma \vdash \varphi_0 \lor \varphi_1$	$\Gamma \vdash \psi$
<u>Γ, φ ⊢ ψ</u> ⊃i	$\frac{\Gamma \vdash \varphi \supset \psi \qquad \Gamma \vdash \varphi}{\Box e} \supset e$
$\Gamma \vdash \varphi \supset \psi$	$\Gamma \vdash \psi$
$\Gamma, \varphi \vdash \neg \psi \qquad \Gamma, \varphi \vdash \psi$	$\Gamma \vdash \neg \neg \varphi$
$\Gamma \vdash \neg \varphi$	$\Gamma \vdash \varphi$

Proof rules for \exists **and** \forall

Introduction rule	Elimination rule
$\frac{\Gamma \vdash \varphi\{y/x\}}{\Gamma \vdash \forall x. \ [\phi]} \forall i \ (y \ fresh)$	$\frac{\Gamma \vdash \forall x. \ [\phi]}{\Gamma \vdash \phi\{t/x\}} \ \forall e$
$\frac{\Gamma \vdash \varphi\{t/x\}}{\Gamma \vdash \exists x. \ [\varphi]} \exists i$	$\frac{\Gamma \vdash \exists x. [\phi] \Gamma, \phi\{y/x\} \vdash \psi}{\Gamma \vdash \psi} \exists e (y \text{ fresh})$

where *t* is a term in the language, and $y \in \mathcal{V}$ is fresh if $y \notin vars(\Gamma \cup \{\varphi, \psi\})$.

$$\frac{1}{\Gamma \vdash \varphi} Ax \quad (\varphi \in \Gamma)$$

We say that $\Gamma \vdash_{\mathscr{G}} \varphi$ if there is a proof of φ from assumptions Γ using Ax and the rules in both the above tables.

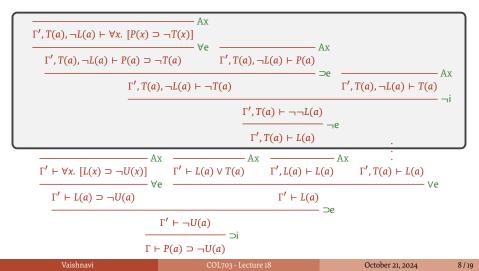
Example o

Show that $\forall x$. $[P(x)] \vdash_{\mathscr{G}} \exists x$. [P(x)]

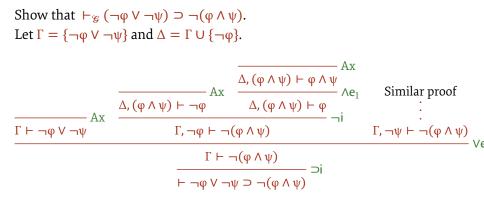
 $\frac{\overline{\forall x. [P(x)] \vdash \forall x. [P(x)]}}{\overline{\forall x. [P(x)] \vdash P(t)}} \stackrel{\text{Ax}}{\forall e} \\ \frac{\forall x. [P(x)] \vdash P(t)}{\overline{\forall x. [P(x)] \vdash \exists x. [P(x)]}} \exists i$

Let $\Gamma = \{ \forall x. [L(x) \supset \neg U(x)], \forall x. [P(x) \supset \neg T(x)], L(a) \lor T(a) \}$. Show that $\Gamma \vdash_{\mathcal{G}} P(a) \supset \neg U(a)$.

Let $\Gamma = \{ \forall x. [L(x) \supset \neg U(x)], \forall x. [P(x) \supset \neg T(x)], L(a) \lor T(a) \}$. Show that $\Gamma \vdash_{\mathscr{G}} P(a) \supset \neg U(a)$. Let $\Gamma' = \Gamma \cup \{P(a)\}$.

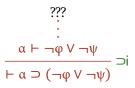


Show that $\vdash_{\mathcal{G}} (\neg \phi \lor \neg \psi) \supset \neg (\phi \land \psi).$



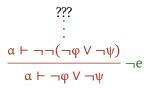
Show that $\vdash_{\mathcal{G}} \neg(\phi \land \psi) \supset (\neg \phi \lor \neg \psi).$

Show that $\vdash_{\mathscr{G}} \neg(\varphi \land \psi) \supset (\neg \varphi \lor \neg \psi)$. Let $\alpha \coloneqq \neg(\varphi \land \psi)$.



- **Tip:** If you have to prove $\Gamma \vdash_{\mathscr{G}} \alpha \lor \beta$, where $\Gamma \nvDash_{\mathscr{G}} \alpha$ and $\Gamma \nvDash_{\mathscr{G}} \beta$, use $\neg e$!
- Same for if you have to prove $\Gamma \vdash_{\mathscr{G}} \exists x. [\alpha]$, but $\Gamma \nvDash_{\mathscr{G}} \alpha(t)$ for any t.
- If all else fails, look to ¬e for help!

Show that $\vdash_{\mathscr{G}} \neg(\varphi \land \psi) \supseteq (\neg \varphi \lor \neg \psi)$. Let $\alpha \coloneqq \neg(\varphi \land \psi)$.



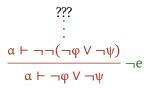
The only way we know to get a "brand new" expression headed by \neg is \neg i! Suppose we had a formula β such that the following held, then done.

$$\frac{\alpha, \neg(\neg \varphi \lor \neg \psi) \vdash \neg \beta \quad \alpha, \neg(\neg \varphi \lor \neg \psi) \vdash \beta}{\frac{\alpha \vdash \neg \neg(\neg \varphi \lor \neg \psi)}{\alpha \vdash \neg \varphi \lor \neg \psi} \neg e} \neg i$$

But what is this β supposed to be?

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Show that $\vdash_{\mathscr{G}} \neg(\varphi \land \psi) \supseteq (\neg \varphi \lor \neg \psi)$. Let $\alpha \coloneqq \neg(\varphi \land \psi)$.



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But what is this β supposed to be? What can we prove from this context?

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Show that $\vdash_{\mathscr{G}} \neg(\phi \land \psi) \supseteq (\neg \phi \lor \neg \psi)$. Let $\alpha \coloneqq \neg(\phi \land \psi)$.

$$\frac{\frac{}{\neg(\neg\varphi \lor \neg\psi), \neg\varphi \vdash \neg\varphi} Ax}{\neg(\neg\varphi \lor \neg\psi), \neg\varphi \vdash \neg\varphi \lor \neg\psi} \lor_{0} \frac{}{\neg(\neg\varphi \lor \neg\psi), \neg\varphi \vdash \neg(\neg\varphi \lor \neg\psi)} Ax}{\neg(\neg\varphi \lor \neg\psi) \vdash \neg\neg\varphi} \neg e$$

Similarly, $\neg(\neg \phi \lor \neg \psi) \vdash \psi$. **Exercise**: Draw this proof tree. Can use Monotonicity and \land i to get a proof π of

Show that $\vdash_{\mathscr{G}} \neg(\phi \land \psi) \supset (\neg \phi \lor \neg \psi)$. Let $\alpha \coloneqq \neg(\phi \land \psi)$.

$$\frac{\overbrace{\alpha,\neg(\neg\varphi \lor \neg\psi)\vdash \alpha}^{\pi} Ax \qquad \vdots \\ a,\neg(\neg\varphi \lor \neg\psi)\vdash(\varphi \land \psi) \\ \frac{\alpha\vdash \neg\neg(\neg\varphi \lor \neg\psi)}{\alpha\vdash \neg\varphi \lor \neg\psi} \neg e$$

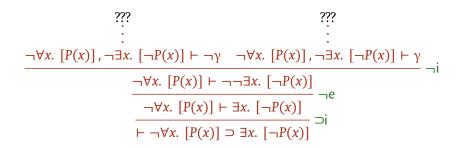
Exercise: Prove that $\forall x$. [$P(x) \lor \neg P(x)$].

Show that $\vdash_{\mathcal{G}} \neg \forall x. [P(x)] \supset \exists x. [\neg P(x)].$

Show that $\vdash_{\mathscr{G}} \neg \forall x. [P(x)] \supset \exists x. [\neg P(x)].$

$$\frac{\overset{???}{\vdots}}{\vdash \neg \forall x. \ [P(x)] \vdash \exists x. \ [\neg P(x)]} \supset i$$

Show that $\vdash_{\mathcal{G}} \neg \forall x. [P(x)] \supset \exists x. [\neg P(x)].$



Show that $\vdash_{\mathcal{G}} \neg \forall x$. $[P(x)] \supset \exists x$. $[\neg P(x)]$. Let $\alpha \coloneqq \neg \forall x$. [P(x)].

$$\frac{\overline{\neg \exists x. [\neg P(x)], \neg P(y) \vdash \neg \exists x. [\neg P(x)]}}{\neg \exists x. [\neg P(x)], \neg P(y) \vdash \neg P(y)} Ax} \xrightarrow{\exists i} \overline{\neg \exists x. [\neg P(x)], \neg P(y) \vdash \exists x. [\neg P(x)]}}_{\neg \exists x. [\neg P(x)] \vdash \neg \neg P(y)} \neg e} \xrightarrow{\neg \exists x. [\neg P(x)] \vdash \neg \neg P(y)}_{\neg \exists x. [\neg P(x)] \vdash \forall x. [P(x)]} \forall i} + Monotonicity$$

$$\frac{a, \neg \exists x. [\neg P(x)] \vdash \neg \forall x. [P(x)]}{\neg \exists x. [\neg P(x)] \vdash \neg \exists x. [\neg P(x)] \vdash \forall x. [P(x)]} \neg i}_{\neg \forall x. [P(x)] \vdash \neg \exists x. [\neg P(x)]} \neg e} \xrightarrow{\neg \forall x. [P(x)] \vdash \exists x. [\neg P(x)]}_{\neg \forall x. [P(x)]} \neg i}_{\neg \forall x. [P(x)] \vdash \exists x. [\neg P(x)]} \neg i}$$

Why move to $\vdash_{\mathcal{G}}$?

- One main reason for moving to ⊢_𝔅 was intuitiveness
- Easier proofs, as we just saw
- Another reason is convenience for automation
- Proof search in $\vdash_{\mathcal{H}}$ is not syntactically decidable (even for PL)
 - Have to search through (infinitely many possible) instantiations of axiom schema which might appear in a proof
- Is ⊢_g better?
- We will see that $\vdash_{\mathscr{G}}$ enjoys some nice properties.
- Monotonicity and cut hold as usual.
- Is there anything that helps with proof search?

Consider a proof of the following sort.

$$\frac{-}{\varphi, \psi \vdash \varphi} \begin{array}{c} Ax & - & Ax \\ \hline \varphi, \psi \vdash \varphi & \varphi, \psi \vdash \psi \\ \hline \hline \frac{\varphi, \psi \vdash \varphi \land \psi}{\varphi, \psi \vdash \varphi} \land e_0 \end{array} \land i$$

Consider a proof of the following sort.

$$\frac{\hline{\phi, \psi \vdash \phi} \quad Ax \quad \hline{\phi, \psi \vdash \psi} \quad Ax}{\frac{\phi, \psi \vdash \phi \land \psi}{\phi, \psi \vdash \phi} \land e_0} \land i$$

We first introduce an Λ , and then immediately eliminate it. Could have replaced this entire proof by the following, smaller proof without any such wasteful detours involving large expressions.

$$\frac{1}{\varphi, \psi \vdash \varphi} Ax$$

- **Exercise**: What does a proof involving a detour on an ∨ or a ⊃ look like?
- Detours on these operators involve the introduction rule for that operator, immediately followed by the elimination rule.
- What about ¬?

- **Exercise**: What does a proof involving a detour on an ∨ or a ⊃ look like?
- Detours on these operators involve the introduction rule for that operator, immediately followed by the elimination rule.
- What about ¬?
- Clearly ¬i followed by ¬e is **not** an unnecessary detour.
- We could not have done the earlier proofs without using this combo!
- However, the expressions we used for $\neg i$ were informed by the context and the expected conclusion.
- Can we eliminate all unnecessary detours?
- Given Γ , α , is there some finite set to which every expression occurring in any proof of $\Gamma \vdash_{\mathscr{C}} \alpha$ belongs?