Lecture 18 - Natural Deduction

Vaishnavi Sundararajan

COL703 - Logic for Computer Science

Recap: Completeness of $\vdash_{HK}$

- **Gödel's Completeness Theorem (1929)**: If $\Gamma \models \varphi$, then $\Gamma \vdash_{HK} \varphi$
- Equivalent statement: *Any consistent set of expressions is satisfiable*
- **Lindenbaum's Lemma**: Every consistent set can be extended to an $\exists$-fulfilled maximally consistent set (MCS).
- Show a model for an $\exists$-fulfilled MCS.
- So every consistent set can be extended to an $\exists$-fulfilled MCS which is satisfiable.
- Same model satisfies the original consistent set (contained) also.

Proof system

- We have shown that $\vdash_{HK}$ is a complete proof system for FOL.
- It is not a particularly intuitive proof system though.
- **Exercise**: Try to prove $\vdash_{HK} \exists x. \; [x \equiv x]$
- Everything has to be cast in terms of $\neg$, $\supset$, and $\forall$
- One needs to know which instances of which axioms to use
- Would like a proof system that more closely mirrors human reasoning
- Fewer axioms, more proof rules!
- Gerhard Gentzen introduced one such, called "Natural Deduction"

Natural Deduction

- No axiom schema, only proof rules
- No need to worry about which instances of which axioms
- Each operator gets an introduction rule and/or an elimination rule
- Introduction rule: Operator appears in the conclusion
- Elimination rule: Operator appears in the (RHS of) premise(s), does not appear in the conclusion

Proof rules for propositional fragment

Introduction rule	Elimination rule
$\dfrac{\Gamma \vdash \varphi_0 \qquad \Gamma \vdash \varphi_1}{\Gamma \vdash \varphi_0 \wedge \varphi_1} \wedge i$	$\dfrac{\Gamma \vdash \varphi_0 \wedge \varphi_1}{\Gamma \vdash \varphi_j} \wedge e_j$
$\dfrac{\Gamma \vdash \varphi_j}{\Gamma \vdash \varphi_0 \vee \varphi_1} \vee i_j$	$\dfrac{\Gamma \vdash \varphi_0 \vee \varphi_1 \qquad \Gamma, \varphi_0 \vdash \psi \qquad \Gamma, \varphi_1 \vdash \psi}{\Gamma \vdash \psi} \vee e$
$\dfrac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \supset \psi} \supset i$	$\dfrac{\Gamma \vdash \varphi \supset \psi \qquad \Gamma \vdash \varphi}{\Gamma \vdash \psi} \supset e$
$\dfrac{\Gamma, \varphi \vdash \neg\psi \qquad \Gamma, \varphi \vdash \psi}{\Gamma \vdash \neg\varphi} \neg i$	$\dfrac{\Gamma \vdash \neg\neg\varphi}{\Gamma \vdash \varphi} \neg e$

Proof rules for $\exists$ and $\forall$

Introduction rule	Elimination rule
$\dfrac{\Gamma \vdash \varphi\{y/x\}}{\Gamma \vdash \forall x.\ [\varphi]}$ $\forall$i (y fresh)	$\dfrac{\Gamma \vdash \forall x.\ [\varphi]}{\Gamma \vdash \varphi\{t/x\}}$ $\forall$e
$\dfrac{\Gamma \vdash \varphi\{t/x\}}{\Gamma \vdash \exists x.\ [\varphi]}$ $\exists$i	$\dfrac{\Gamma \vdash \exists x.\ [\varphi] \quad \Gamma, \varphi\{y/x\} \vdash \psi}{\Gamma \vdash \psi}$ $\exists$e (y fresh)

where t is a term in the language, and $y \in \mathscr{V}$ is fresh if $y \notin \text{vars}(\Gamma \cup \{\varphi, \psi\})$.

$$\frac{}{\Gamma \vdash \varphi}\ \text{Ax} \quad (\varphi \in \Gamma)$$

We say that $\Gamma \vdash_{\mathscr{G}} \varphi$ if there is a proof of φ from assumptions Γ using Ax and the rules in both the above tables.

Example 0

Show that $\forall x.\ [P(x)] \vdash_{\mathcal{G}} \exists x.\ [P(x)]$

$$\dfrac{\dfrac{\dfrac{}{\forall x.\ [P(x)] \vdash \forall x.\ [P(x)]}\ \text{Ax}}{\forall x.\ [P(x)] \vdash P(t)}\ \forall\text{e}}{\forall x.\ [P(x)] \vdash \exists x.\ [P(x)]}\ \exists\text{i}$$

Example 1

Let $\Gamma = \{\forall x.\ [L(x) \supset \neg U(x)], \forall x.\ [P(x) \supset \neg T(x)], L(a) \vee T(a)\}$. Show that $\Gamma \vdash_{\mathcal{G}} P(a) \supset \neg U(a)$.

Example 1

Let $\Gamma = \{\forall x.\ [L(x) \supset \neg U(x)], \forall x.\ [P(x) \supset \neg T(x)], L(a) \vee T(a)\}$. Show that $\Gamma \vdash_{\mathcal{G}} P(a) \supset \neg U(a)$. Let $\Gamma' = \Gamma \cup \{P(a)\}$.

$$\dfrac{}{\Gamma', T(a), \neg L(a) \vdash \forall x.\ [P(x) \supset \neg T(x)]}\ \text{Ax}$$

$$\dfrac{\Gamma', T(a), \neg L(a) \vdash \forall x.\ [P(x) \supset \neg T(x)]}{\Gamma', T(a), \neg L(a) \vdash P(a) \supset \neg T(a)}\ \forall\text{e} \qquad \dfrac{}{\Gamma', T(a), \neg L(a) \vdash P(a)}\ \text{Ax}$$

$$\dfrac{\Gamma', T(a), \neg L(a) \vdash P(a) \supset \neg T(a) \qquad \Gamma', T(a), \neg L(a) \vdash P(a)}{\Gamma', T(a), \neg L(a) \vdash \neg T(a)}\ \supset\text{e} \qquad \dfrac{}{\Gamma', T(a), \neg L(a) \vdash T(a)}\ \text{Ax}$$

$$\dfrac{\Gamma', T(a), \neg L(a) \vdash \neg T(a) \qquad \Gamma', T(a), \neg L(a) \vdash T(a)}{\Gamma', T(a) \vdash \neg\neg L(a)}\ \neg\text{i}$$

$$\dfrac{\Gamma', T(a) \vdash \neg\neg L(a)}{\Gamma', T(a) \vdash L(a)}\ \neg\text{e}$$

$$\dfrac{}{\Gamma' \vdash \forall x.\ [L(x) \supset \neg U(x)]}\ \text{Ax} \qquad \dfrac{}{\Gamma' \vdash L(a) \vee T(a)}\ \text{Ax} \qquad \dfrac{}{\Gamma', L(a) \vdash L(a)}\ \text{Ax} \qquad \begin{matrix}\vdots \\ \Gamma', T(a) \vdash L(a)\end{matrix}$$

$$\dfrac{\Gamma' \vdash \forall x.\ [L(x) \supset \neg U(x)]}{\Gamma' \vdash L(a) \supset \neg U(a)}\ \forall\text{e} \qquad \dfrac{\Gamma' \vdash L(a) \vee T(a) \qquad \Gamma', L(a) \vdash L(a) \qquad \Gamma', T(a) \vdash L(a)}{\Gamma' \vdash L(a)}\ \vee\text{e}$$

$$\dfrac{\Gamma' \vdash L(a) \supset \neg U(a) \qquad \Gamma' \vdash L(a)}{\Gamma' \vdash \neg U(a)}\ \supset\text{e}$$

$$\dfrac{\Gamma' \vdash \neg U(a)}{\Gamma \vdash P(a) \supset \neg U(a)}\ \supset\text{i}$$

Example 2

Show that $\vdash_{\mathcal{G}} (\neg\varphi \vee \neg\psi) \supset \neg(\varphi \wedge \psi)$.

Example 2

Show that $\vdash_{\mathscr{G}} (\neg\varphi \lor \neg\psi) \supset \neg(\varphi \land \psi)$.
Let $\Gamma = \{\neg\varphi \lor \neg\psi\}$ and $\Delta = \Gamma \cup \{\neg\varphi\}$.

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \neg\varphi \lor \neg\psi}\,\text{Ax} \qquad \dfrac{\dfrac{}{\Delta, (\varphi \land \psi) \vdash \neg\varphi}\,\text{Ax} \quad \dfrac{\dfrac{}{\Delta, (\varphi \land \psi) \vdash \varphi \land \psi}\,\text{Ax}}{\Delta, (\varphi \land \psi) \vdash \varphi}\,\land e_1}{\Gamma, \neg\varphi \vdash \neg(\varphi \land \psi)}\,\neg i \qquad \begin{array}{c}\text{Similar proof}\\ \vdots \\ \Gamma, \neg\psi \vdash \neg(\varphi \land \psi)\end{array}}{\Gamma \vdash \neg(\varphi \land \psi)}\,\lor e}{\vdash \neg\varphi \lor \neg\psi \supset \neg(\varphi \land \psi)}\,\supset i$$

Example 3

Show that $\vdash_{\mathscr{G}} \neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$.

Example 3

Show that $\vdash_{\mathscr{G}} \neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$. Let $\alpha \coloneqq \neg(\varphi \wedge \psi)$.

$$\dfrac{\begin{array}{c} ??? \\ \vdots \\ \alpha \vdash \neg\varphi \vee \neg\psi \end{array}}{\vdash \alpha \supset (\neg\varphi \vee \neg\psi)} \supset\mathrm{i}$$

- **Tip:** If you have to prove $\Gamma \vdash_{\mathscr{G}} \alpha \vee \beta$, where $\Gamma \nvdash_{\mathscr{G}} \alpha$ and $\Gamma \nvdash_{\mathscr{G}} \beta$, use $\neg$e!
- Same for if you have to prove $\Gamma \vdash_{\mathscr{G}} \exists x.\ [\alpha]$, but $\Gamma \nvdash_{\mathscr{G}} \alpha(t)$ for any t.
- If all else fails, look to $\neg$e for help!

Example 3

Show that $\vdash_{\mathcal{G}} \neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$. Let $\alpha := \neg(\varphi \wedge \psi)$.

$$\dfrac{\begin{array}{c} ??? \\ \vdots \\ \alpha \vdash \neg\neg(\neg\varphi \vee \neg\psi) \end{array}}{\alpha \vdash \neg\varphi \vee \neg\psi}\ \neg e$$

The only way we know to get a "brand new" expression headed by $\neg$ is $\neg i$!

Suppose we had a formula β such that the following held, then done.

$$\dfrac{\dfrac{\begin{array}{c}\vdots \\ \alpha, \neg(\neg\varphi \vee \neg\psi) \vdash \neg\beta\end{array} \quad \begin{array}{c}\vdots \\ \alpha, \neg(\neg\varphi \vee \neg\psi) \vdash \beta\end{array}}{\alpha \vdash \neg\neg(\neg\varphi \vee \neg\psi)}\ \neg i}{\alpha \vdash \neg\varphi \vee \neg\psi}\ \neg e$$

But what is this β supposed to be?

Example 3

Show that $\vdash_{\mathcal{G}} \neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$. Let $\alpha \coloneqq \neg(\varphi \wedge \psi)$.

$$\dfrac{\begin{array}{c}???\\ \vdots \\ \alpha \vdash \neg\neg(\neg\varphi \vee \neg\psi)\end{array}}{\alpha \vdash \neg\varphi \vee \neg\psi}\ \neg e$$

The only way we know to get a "brand new" expression headed by $\neg$ is $\neg i$!

Suppose we had a formula β such that the following held, then done.

$$\dfrac{\dfrac{\begin{array}{c}\vdots \\ \alpha, \neg(\neg\varphi \vee \neg\psi) \vdash \neg\beta\end{array} \quad \begin{array}{c}\vdots \\ \alpha, \neg(\neg\varphi \vee \neg\psi) \vdash \beta\end{array}}{\alpha \vdash \neg\neg(\neg\varphi \vee \neg\psi)}\ \neg i}{\alpha \vdash \neg\varphi \vee \neg\psi}\ \neg e$$

But what is this β supposed to be? What can we prove from this context?

Example 3

Show that $\vdash_{\mathscr{G}} \neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$. Let $\alpha \coloneqq \neg(\varphi \wedge \psi)$.

$$\dfrac{\dfrac{\dfrac{}{\neg(\neg\varphi \vee \neg\psi), \neg\varphi \vdash \neg\varphi}\,\text{Ax}}{\neg(\neg\varphi \vee \neg\psi), \neg\varphi \vdash \neg\varphi \vee \neg\psi}\,\vee i_0 \quad \dfrac{}{\neg(\neg\varphi \vee \neg\psi), \neg\varphi \vdash \neg(\neg\varphi \vee \neg\psi)}\,\text{Ax}}{\dfrac{\neg(\neg\varphi \vee \neg\psi) \vdash \neg\neg\varphi}{\neg(\neg\varphi \vee \neg\psi) \vdash \varphi}\,\neg e}\,\neg i$$

Similarly, $\neg(\neg\varphi \vee \neg\psi) \vdash \psi$. **Exercise**: Draw this proof tree.

Can use Monotonicity and $\wedge i$ to get a proof π of

$$\alpha, \neg(\neg\varphi \vee \neg\psi) \vdash_{\mathscr{G}} \varphi \wedge \psi$$

Example 3

Show that $\vdash_{\mathscr{G}} \neg(\varphi \land \psi) \supset (\neg\varphi \lor \neg\psi)$. Let $\alpha \coloneqq \neg(\varphi \land \psi)$.

$$
\dfrac{\dfrac{}{\alpha, \neg(\neg\varphi \lor \neg\psi) \vdash \alpha}\,\text{Ax} \qquad \begin{matrix}\pi \\ \vdots \\ \alpha, \neg(\neg\varphi \lor \neg\psi) \vdash (\varphi \land \psi)\end{matrix}}{\dfrac{\alpha \vdash \neg\neg(\neg\varphi \lor \neg\psi)}{\alpha \vdash \neg\varphi \lor \neg\psi}\,\neg\text{e}}\,\neg\text{i}
$$

Exercise: Prove that $\forall x.\ [P(x) \lor \neg P(x)]$.

Example 4

Show that $\vdash_{\mathscr{G}} \neg\forall x.\ [P(x)] \supset \exists x.\ [\neg P(x)]$.

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Show that $\vdash_{\mathcal{G}} \neg\forall x.\ [P(x)] \supset \exists x.\ [\neg P(x)]$.

$$\dfrac{\begin{matrix}???\\ \vdots \\ \neg\forall x.\ [P(x)] \vdash \exists x.\ [\neg P(x)]\end{matrix}}{\vdash \neg\forall x.\ [P(x)] \supset \exists x.\ [\neg P(x)]}\ \supset\mathrm{i}$$

Example 4

Show that $\vdash_{\mathscr{G}} \neg\forall x.\ [P(x)] \supset \exists x.\ [\neg P(x)]$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{???}{\vdots}}{\neg\forall x.\ [P(x)], \neg\exists x.\ [\neg P(x)] \vdash \neg\gamma} \quad \dfrac{\dfrac{???}{\vdots}}{\neg\forall x.\ [P(x)], \neg\exists x.\ [\neg P(x)] \vdash \gamma}}{\neg\forall x.\ [P(x)] \vdash \neg\neg\exists x.\ [\neg P(x)]}\ \neg\mathrm{i}}{\dfrac{\neg\forall x.\ [P(x)] \vdash \exists x.\ [\neg P(x)]}{\vdash \neg\forall x.\ [P(x)] \supset \exists x.\ [\neg P(x)]}\ \supset\mathrm{i}}\ \neg\mathrm{e}
$$

Example 4

Show that $\vdash_{\mathscr{G}} \neg\forall x.\ [P(x)] \supset \exists x.\ [\neg P(x)]$. Let $\alpha \coloneqq \neg\forall x.\ [P(x)]$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg\exists x.\ [\neg P(x)], \neg P(y) \vdash \neg\exists x.\ [\neg P(x)]}\ \text{Ax} \qquad \dfrac{\dfrac{}{\neg\exists x.\ [\neg P(x)], \neg P(y) \vdash \neg P(y)}\ \text{Ax}}{\neg\exists x.\ [\neg P(x)], \neg P(y) \vdash \exists x.\ [\neg P(x)]}\ \exists\text{i}}{\neg\exists x.\ [\neg P(x)] \vdash \neg\neg P(y)}\ \neg\text{i}}{\neg\exists x.\ [\neg P(x)] \vdash P(y)}\ \neg\text{e}}{\neg\exists x.\ [\neg P(x)] \vdash \forall x.\ [P(x)]}\ \forall\text{i}$$

+ Monotonicity

$\vdots$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\alpha, \neg\exists x.\ [\neg P(x)] \vdash \neg\forall x.\ [P(x)]}\ \text{Ax} \qquad \alpha, \neg\exists x.\ [\neg P(x)] \vdash \forall x.\ [P(x)]}{\neg\forall x.\ [P(x)] \vdash \neg\neg\exists x.\ [\neg P(x)]}\ \neg\text{i}}{\neg\forall x.\ [P(x)] \vdash \exists x.\ [\neg P(x)]}\ \neg\text{e}}{\vdash \neg\forall x.\ [P(x)] \supset \exists x.\ [\neg P(x)]}\ \supset\text{i}$$

Why move to $\vdash_{\mathcal{G}}$?

- One main reason for moving to $\vdash_{\mathcal{G}}$ was intuitiveness
- Easier proofs, as we just saw
- Another reason is convenience for automation
- Proof search in $\vdash_{\mathcal{H}}$ is not syntactically decidable (even for PL)
  - Have to search through (infinitely many possible) instantiations of axiom schema which might appear in a proof
- Is $\vdash_{\mathcal{G}}$ better?
- We will see that $\vdash_{\mathcal{G}}$ enjoys some nice properties.
- Monotonicity and cut hold as usual.
- Is there anything that helps with proof search?

Unnecessary detours in proofs

Consider a proof of the following sort.

$$\dfrac{\dfrac{\dfrac{}{\varphi, \psi \vdash \varphi}\,\text{Ax} \qquad \dfrac{}{\varphi, \psi \vdash \psi}\,\text{Ax}}{\varphi, \psi \vdash \varphi \wedge \psi}\,\wedge\text{i}}{\varphi, \psi \vdash \varphi}\,\wedge\text{e}_0$$

Unnecessary detours in proofs

Consider a proof of the following sort.

$$\dfrac{\dfrac{}{\varphi, \psi \vdash \varphi}\text{Ax} \qquad \dfrac{}{\varphi, \psi \vdash \psi}\text{Ax}}{\dfrac{\varphi, \psi \vdash \varphi \land \psi}{\varphi, \psi \vdash \varphi}\land e_0}\land i$$

We first introduce an $\land$, and then immediately eliminate it.

Could have replaced this entire proof by the following, smaller proof without any such wasteful detours involving large expressions.

$$\dfrac{}{\varphi, \psi \vdash \varphi}\text{Ax}$$

Unnecessary detours in proofs

- **Exercise**: What does a proof involving a detour on an $\vee$ or a $\supset$ look like?
- Detours on these operators involve the introduction rule for that operator, immediately followed by the elimination rule.
- What about $\neg$?

Unnecessary detours in proofs

- **Exercise**: What does a proof involving a detour on an $\vee$ or a $\supset$ look like?
- Detours on these operators involve the introduction rule for that operator, immediately followed by the elimination rule.
- What about $\neg$?
- Clearly $\neg i$ followed by $\neg e$ is **not** an unnecessary detour.
- We could not have done the earlier proofs without using this combo!
- However, the expressions we used for $\neg i$ were informed by the context and the expected conclusion.
- Can we eliminate all unnecessary detours?
- Given Γ, α, is there some finite set to which every expression occurring in any proof of $\Gamma \vdash_{\mathscr{G}} \alpha$ belongs?