Lecture 16 - FO Completeness

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Recap: FO Resolution

- **Substitution Lemma**: Given an interpretation $\mathcal{I} = ((M, \iota), \sigma)$, an expression $\varphi \in \mathsf{FO}_\Sigma$, and a substitution $\{u/x\}$ such that $u^{\mathcal{I}} = m \in M$, $\mathcal{I} \models \varphi\{u/x\}$ iff $\mathcal{I}[x \mapsto m] \models \varphi$.
- Let δ_1, δ_2 be clauses s.t. $\mathsf{fv}(\delta_1) \cap \mathsf{fv}(\delta_2) = \emptyset$
- Let $P \in \mathscr{P}$ be a k-ary predicate symbol
- Let $L_1 = \left\{P(u_1, \dots, u_k) \in \delta_1 \,\middle|\, u_1, \dots, u_k \in \mathrm{T}(\Sigma)\right\}$ such that $\delta_1 = \delta_1' \cup L_1$
- Let $L_2 = \left\{\neg P(v_1, \dots, v_k) \in \delta_2 \,\middle|\, v_1, \dots, v_k \in \mathrm{T}(\Sigma)\right\}$ such that $\delta_2 = \delta_2' \cup L_2$
- Denote by $\overline{L}_2$ the set $\{P(v_1, \dots, v_k) \in \delta_2 \mid v_1, \dots, v_k \in \mathrm{T}(\Sigma)\}$
- Let $L_1 \cup \overline{L}_2$ be unifiable, with θ an mgu

$$\frac{\delta_1' \cup L_1 \qquad \delta_2' \cup L_2}{\theta(\delta_1' \cup \delta_2')}\ \theta$$

Towards a proof system

- The resolution procedure (linking unsatisfiability to the derivation of an empty clause) is sound and complete
- However, this rule does not provide a complete proof system for first-order logic (**Exercise**: Think about why!)
- Move to a less minimal proof system (which might be complete)
- $\{\neg, \supset, \forall\}$ is a functionally complete set of operators for FOL
- Can extend $\vdash_{\mathcal{H}}$ to get $\vdash_{HK}$ for FOL
- All FO expression instances of PL tautologies are valid
- A **generalization** of an expression φ is any $\forall x_1 \dots x_n.\ \varphi$, where $n \geqslant 0$.

FO syntax: $\varphi, \psi, \chi \coloneqq t_1 \equiv t_2 \mid P(t_1, \dots, t_n) \mid \neg\psi \mid \psi \supset \chi \mid \forall x.\ [\psi]$

System $\vdash_{HK}$ for FO

All generalizations of the following, along with MP.

(H1a) $\varphi \supset (\psi \supset \varphi)$

(H1b) $(\varphi \supset (\psi \supset \chi)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \chi))$

(H1c) $(\neg\varphi \supset \neg\psi) \supset ((\neg\varphi \supset \psi) \supset \varphi)$

(H2a) $x \equiv x$

(H2b) $x \equiv y \supset \big((\varphi(x) \supset \varphi(y)) \wedge (\varphi(y) \supset \varphi(x))\big)$

(H3a) $\forall x.\ [\varphi \supset \psi] \supset (\forall x.\ [\varphi] \supset \forall x.\ [\psi])$

(H3b) $\varphi \supset \forall x.\ [\varphi]$ where x does not appear free in φ

(H3c) $\forall x.\ [\varphi] \supset \varphi\{t/x\}$ for any term t

$$\frac{\varphi \supset \psi \qquad \varphi}{\psi}\ \text{MP}$$

We denote provability in this system with the symbol $\vdash_{HK}$.

Soundness of $\vdash_{HK}$

- **Theorem (Soundness)**: If $\vdash_{HK} \varphi$, then $\vDash \varphi$
- Show that axioms are valid, and that MP preserves validity
- Might need the following lemma: For every φ, every $x \in \mathscr{V}$, and every $y \notin \mathsf{vars}(\forall x.\ [\varphi])$,
  $\vDash (\forall x.\ [\varphi] \supset \forall y.\ [\varphi\{y/x\}]) \wedge (\forall y.\ [\varphi\{y/x\}] \supset \forall x.\ [\varphi])$.
- For all Γ, α, β, we have the following
- **Deduction Theorem**: $\Gamma, \alpha \vdash_{HK} \beta$ iff $\Gamma \vdash_{HK} \alpha \supset \beta$.
- **Cut** is admissible: If $\Gamma \vdash \alpha$ and $\Gamma, \alpha \vdash \beta$, then $\Gamma \vdash \beta$.
- **Lemma (Replacement by new variables)**: Suppose $\Gamma \vdash_{HK} \varphi$, and $y \in \mathscr{V} \setminus (\mathsf{vars}(\Gamma) \cup \mathsf{vars}(\varphi))$. Then, $\Gamma\{y/x\} \vdash_{HK} \varphi\{y/x\}$.
- **Exercise**: Prove all these statements.

Substituting bound variables: Equivalence

Lemma: For every φ, every $x \in \mathscr{V}$, and every $y \notin \mathsf{vars}(\forall x.\ [\varphi])$,
$\vdash_{HK} (\forall x.\ [\varphi] \supset \forall y.\ [\varphi\{y/x\}]) \wedge (\forall y.\ [\varphi\{y/x\}] \supset \forall x.\ [\varphi])$.

Proof: Enough to show **(⇒)**, i.e. $\vdash_{HK} \forall x.\ [\varphi] \supset \forall y.\ [\varphi\{y/x\}]$.
(⇐) follows since $x \notin \mathsf{vars}(\forall y.\ [\varphi\{y/x\}])$, and $\varphi\{y/x\}\{x/y\} = \varphi$.

$$\dfrac{\dfrac{}{\forall y.\ [\forall x.\ [\varphi] \supset \varphi\{y/x\}] \supset \big(\forall y.\ [\forall x.\ [\varphi]] \supset \forall y.\ [\varphi\{y/x\}]\big)}\,\text{H3a} \qquad \dfrac{}{\forall y.\ [\forall x.\ [\varphi] \supset \varphi\{y/x\}]}\,\text{H3c+G}}{\forall y.\ [\forall x.\ [\varphi]] \supset \forall y.\ [\varphi\{y/x\}]}\,\text{MP} \quad (\pi)$$

$$\dfrac{\dfrac{}{\forall x.\ [\varphi] \supset \forall y.\ [\forall x.\ [\varphi]]}\,\text{H3b} \qquad \begin{array}{c}\pi\\ \vdots\\ \forall y.\ [\forall x.\ [\varphi]] \supset \forall y.\ [\varphi\{y/x\}]\end{array}}{\forall x.\ [\varphi] \supset \forall y.\ [\varphi\{y/x\}]}\,\text{Cut}$$

Universal generalization lemma

Lemma (Universal generalization): Suppose $\Gamma \vdash_{HK} \varphi\{y/x\}$, where $y \notin \mathsf{fv}(\Gamma) \cup \mathsf{fv}(\varphi)$. Then, $\Gamma \vdash_{HK} \forall x.\ [\varphi]$.

Proof: Suppose $\Gamma \vdash_{HK} \varphi\{y/x\}$ via a proof π. We first show the following:

For any sequent $\Gamma \vdash_{HK} \alpha_i$ *appearing in the proof* π, $\Gamma \vdash_{HK} \forall y.\ [\alpha_i]$.

(Then, $\Gamma \vdash_{HK} \forall y.\ [\varphi\{y/x\}]$, and by the previous lemma, $\Gamma \vdash_{HK} \forall x.\ [\varphi]$.)

The proof is by induction on the structure of π.

Base case(s): Suppose α_i is an instance of an axiom. Then, $\forall x.\ [\alpha_i]$ is a generalization of an axiom, and hence, also an axiom. Otherwise, suppose $\alpha_i \in \Gamma$. Then, $y \notin \mathsf{fv}(\alpha_i)$. Thus, **(H3b)** gives us $\alpha_i \supset \forall y.\ [\alpha_i]$.

$$\dfrac{\dfrac{}{\Gamma \vdash \alpha_i \supset \forall y.\ [\alpha_i]}\ \text{H3b} \qquad \dfrac{}{\Gamma \vdash \alpha_i}\ \text{Ax}}{\Gamma \vdash \forall y.\ [\alpha_i]}\ \text{MP}$$

Universal generalization lemma: Proof

Induction case: α_i is obtained by applying MP to some $\alpha_j \supset \alpha_i$ and α_j, both appearing in shorter subtrees. By IH, $\Gamma \vdash \forall y.\ [\alpha_j \supset \alpha_i]$ and $\Gamma \vdash \forall y.\ [\alpha_j]$.

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \forall y.\ [\alpha_j \supset \alpha_i] \supset (\forall y.\ [\alpha_j] \supset \forall y.\ [\alpha_i])}\ \text{H3a} \qquad \begin{array}{c}\text{IH}\\ \vdots \\ \Gamma \vdash \forall y.\ [\alpha_j \supset \alpha_i]\end{array}}{\Gamma \vdash \forall y.\ [\alpha_j] \supset \forall y.\ [\alpha_i]}\ \text{MP} \qquad \begin{array}{c}\text{IH}\\ \vdots \\ \Gamma \vdash \forall y.\ [\alpha_j]\end{array}}{\Gamma \vdash \forall y.\ [\alpha_i]}$$

Completeness of $\vdash_{HK}$

- **Gödel's Completeness Theorem (1929)**: If $\Gamma \vDash \varphi$, then $\Gamma \vdash_{HK} \varphi$
- Want a slightly different, equivalent formulation of this statement
- Introduce a notion of **consistency**
- An expression φ is said to be **consistent** if $\nvdash_{HK} \neg\varphi$
- A finite set $\{\varphi_1, \ldots, \varphi_n\}$ is consistent if $\bigwedge_{1 \leqslant i \leqslant n} \varphi_i$ is consistent
- An arbitrary set Γ is consistent if each of its finite subsets is consistent.
- Equivalent statement: *Any consistent set of expressions is satisfiable*
- **Exercise**: Show that this is equivalent to the Completeness statement.

Completeness of $\vdash_{HK}$

- Suppose we start out with a consistent set of expressions Γ
- The proof becomes easier if we can assume $\mathscr{V} \setminus \mathsf{vars}(\Gamma)$ to be infinite.
- We achieve this as follows. Let $\mathscr{V} = \{x_0, x_1, x_2, \ldots\}$
- Partition this set into $\mathscr{V}_e = \{x_0, x_2, x_4, \ldots\}$ and $\mathscr{V}_o = \{x_1, x_3, x_5, \ldots\}$
- Given a Γ, form Δ by systematically replacing each occurrence (free or bound) of x_i in Γ by x_{2i} for all $i \geqslant 0$.
- $\mathsf{vars}(\Delta) \subseteq \mathscr{V}_e$, so $\mathscr{V} \setminus \mathsf{vars}(\Delta)$ is infinite.
- We now need to prove the following:
  - If Γ is consistent, then Δ is consistent
  - If Δ is satisfiable, then Γ is satisfiable
- Once we prove these, we can assume $\mathscr{V} \setminus \mathsf{vars}(\Gamma)$ to be infinite in the rest of the presentation.

Γ consistent $\Rightarrow \Delta$ consistent

- Proof by contradiction. Suppose Δ is inconsistent.
- Then, there is a $\{\delta_1, \dots, \delta_k\} \subseteq_{\text{fin}} \Delta$ such that $\vdash_{HK} \neg(\delta_1 \wedge \dots \wedge \delta_k)$
- Let n be such that $i < 2n$ for every i where $x_i \in \mathsf{fv}(\bigcup_{1 \leqslant j \leqslant k} \delta_j)$.
- Replace every $x_{2j} \in \mathsf{vars}(\bigcup_{1 \leqslant j \leqslant k} \delta_j)$ by x_{2n+j} to get $\{\rho_1, \dots, \rho_k\}$
- **Claim**: $\vdash_{HK} \neg(\rho_1 \wedge \dots \wedge \rho_k)$ **Exercise**: Prove this claim.
- Replace every x_{2n+j} by x_j to get $\{\gamma_1, \dots, \gamma_k\} \subseteq_{\text{fin}} \Gamma$
- $\vdash_{HK} \neg(\gamma_1 \wedge \dots \wedge \gamma_k)$
- Thus, Γ is inconsistent.

Δ satisfiable $\Rightarrow$ Γ satisfiable

- Suppose $(\mathscr{M}, \sigma) \models \Delta$.
- Only variables from $\mathscr{V}_e$ appear in Δ
- We replace every occurrence of x_{2i} by x_i to get Γ
- $(\mathscr{M}, \sigma') \models \Gamma$, where $\sigma'(x_i) = \sigma(x_{2i})$
- Thus, if Δ is satisfiable, then so is Γ

Lindenbaum's Lemma

- A set Γ is maximally consistent if Γ is consistent, and $\Gamma \cup \{\varphi\}$ is inconsistent for any FO expression $\varphi \notin \Gamma$.
- A set Γ is said to be $\exists$-fulfilled iff for every expression of the form $\neg\forall x.\ [\alpha] \in \Gamma$, there exists some term t such that $\neg\alpha\{t/x\} \in \Gamma$.
- **Lindenbaum's Lemma**: Every consistent set can be extended to an $\exists$-fulfilled MCS.
- Given a consistent Γ, we build an $\exists$-fulfilled MCS which extends Γ.
- As earlier, fix an enumeration of expressions, and examine each.

Lindenbaum's Lemma: Proof

- Fix an enumeration $\varphi_0, \varphi_1, \varphi_2, \ldots$ of the expressions in FO_Σ
- Also fix an enumeration $x_0, x_1, x_2, \ldots$ of the variables in $\mathscr{V}$
- Now, we build the following sequence $\Gamma_0, \Gamma_1, \ldots$ of sets of formulas.
- $\Gamma_0 := \Gamma$, and for every $i \geqslant 0$,

$$\Gamma_{i+1} := \begin{cases} \Gamma_i' & \text{if } \Gamma_i' \text{ consistent and } \varphi_i \text{ not of the form } \neg\forall x.\ [\alpha] \\ \Gamma_i' \cup \{\neg\alpha\{y/x\}\} & \text{if } \Gamma_i' \text{ consistent, } \varphi_i = \neg\forall x.\ [\alpha]\text{, and} \\ & \quad y \text{ the first variable not in } \mathsf{fv}(\Gamma_i) \cup \mathsf{vars}(\varphi_i)^1 \\ \Gamma_i & \text{if } \Gamma_i' \text{ not consistent} \end{cases}$$

  where $\Gamma_i' = \Gamma_i \cup \{\varphi_i\}$.
- Finally, $\Gamma_{ext} := \bigcup_{i \geqslant 0} \Gamma_i$

[1] We can get away with only requiring that y is the first variable not in $\mathsf{fv}(\Gamma_i) \cup \mathsf{fv}(\alpha)$ as long as we somehow ensure that $y \notin \mathsf{bv}(\alpha)$

Lindenbaum's Lemma: Proof

- **Claim**: Γ_{ext} is maximally consistent and $\exists$-fulfilled.
- We first show that each Γ_i is consistent (by induction on i)
- **Base case**: $\Gamma_0 = \Gamma$, consistent by assumption.
- **Induction step**: Suppose Γ_i is consistent. Two cases arise: Either $\Gamma_i' = \Gamma_i \cup \{\varphi_i\}$ is consistent or not.
- In the latter case, $\Gamma_{i+1} = \Gamma_i$, and Γ_{i+1} is also consistent.
- If $\Gamma_i' = \Gamma_i \cup \{\varphi_i\}$ is consistent, and if φ_i is not of the form $\neg\forall x.\ [\alpha]$, then $\Gamma_{i+1} = \Gamma_i'$, so consistent by construction.

Lindenbaum's Lemma: Proof

- If $\varphi_i = \neg\forall x.\ [\alpha]$ for some α, and $\Gamma_i \cup \{\neg\forall x.\ [\alpha]\}$ is consistent , we set $\Gamma_{i+1} = \Gamma_i \cup \{\neg\forall x.\ [\alpha], \neg\alpha\{y/x\}\}$, where y is the first variable not in $\mathsf{fv}(\Gamma_i) \cup \mathsf{vars}(\varphi_i)$
- Suppose towards a contradiction that Γ_{i+1} is not consistent
- There is $\{\gamma_1, \dots, \gamma_k\} \subseteq_{\text{fin}} \Gamma_i$ such that $\neg\forall x.\ [\alpha], \gamma_1, \dots, \gamma_k \vdash \alpha\{y/x\}$. **Why?**
- Since $y \notin \mathsf{fv}(\Gamma_i) \cup \mathsf{vars}(\varphi_i)$, we can use Universal Generalization to get $\neg\forall x.\ [\alpha], \gamma_1, \dots, \gamma_k \vdash \forall x.\ [\alpha]$.
- One can avoid using $\neg\varphi$ as an assumption to prove φ for any φ. So $\gamma_1, \dots, \gamma_k \vdash \forall x.\ [\alpha]$
- But this contradicts the fact that $\Gamma_i \cup \{\neg\forall x.\ [\alpha]\}$ is consistent!
- So Γ_{i+1} is consistent for every i.

Lindenbaum's Lemma: Proof

- Γ_{ext} is consistent, since each finite subset of Γ_{ext} is also a finite subset of Γ_i for some $i \geqslant 0$. **Exercise:** Why only one Γ_i and not multiple?
- For every φ_ℓ such that $\Gamma_{\text{ext}} \cup \{\varphi_\ell\}$ is consistent, $\Gamma_\ell \cup \varphi_\ell$ is also consistent (reasoning as above), so $\varphi_\ell \in \Gamma_{\ell+1} \subseteq \Gamma_{\text{ext}}$. Therefore, Γ_{ext} is maximally consistent.
- Consider $\varphi_\ell = \neg\forall x.\ [\alpha] \in \Gamma_{\text{ext}}$. Note that $\Gamma_\ell \cup \{\varphi_\ell\}$ is consistent (as above). So $\neg\alpha\{y/x\} \in \Gamma_{\ell+1} \subseteq \Gamma_{\text{ext}}$ for some y, by construction. Therefore, Γ_{ext} is also $\exists$-fulfilled.
- Thus, we have shown that every consistent set Γ can be extended to an $\exists$-fulfilled MCS Γ_{ext}.