COL352 Problem Sheet 7

Topics: Undecidability and Reductions

Problem 1. (Easy) Show that the collection of decidable languages is closed under the operation of

1. union
2. complementation
3. concatenation
4. intersection
5. star

Problem 2. (Easy) Show that the collection of Turing-recognizable languages is closed under the operation of

1. union
2. concatenation
3. intersection
4. star

Problem 3. (Medium) Let the language $MEM_\epsilon = \{\langle G \rangle |\ G \text{ is a CFG that generates } \epsilon\}$, where ϵ is the empty string. Show that MEM_ϵ is decidable.

Problem 4. (Easy) Show that C is a Turing recognizable language $\iff$ there exists a decidable language D such that $C = \{x \mid \exists y \langle x, y \rangle \in D\}$.

Problem 5. (Medium) Show that the following language is undecidable:

$$L = \{\langle M \rangle \# w\}$$

such that M is a single tape TM with singly infinite tape (i.e., tape has a left cell) and the run of M on input w tries to move left on the leftmost cell at some point during the run.

Show that the language L is undecidable.

Problem 6. (Hard) Show that the following language is decidable:

$$L = \{\langle M \rangle \# w\}$$

such that M is a TM whose tape head never moves left during the run on input w.

Hint: If head is only moving right, then can you analyze how long it takes to read full input?

Problem 7. (Hard) The *Busy Beaver* [1] function BB(n) is defined for an n-state Turing machine (TM) with a two-symbol alphabet $\{0, 1\}$ and a single infinite tape. A TM is an n-state Busy Beaver candidate if it halts when started on a blank tape (all 0s). BB(n) is the maximum number of state transitions by any such halting TM. For $n = 1$, a 1-state TM (states $\{q_1, h\}$) can, at best, undergo one transition and halt, e.g., via $\delta(q_1, 0) = (h, 1, R)$, producing BB(1) = 1, as other configurations either loop or write on the tape without any state transition.

Show that no Turing Machine can compute the value of BB(n) for all $n \geq 1$.

Problem 8. (Very Hard) Post correspondence problem is defined as below: Given two lists of strings over an alphabet Σ (not unary):

$$A = w_1, w_2, \ldots, w_k \quad \text{and} \quad B = x_1, x_2, \ldots, x_k$$

Does there exist a sequence of indices $i_1, i_2, \ldots, i_n$ (where $1 \leq i_j \leq k$) such that:

$$w_{i_1} w_{i_2} \ldots w_{i_n} = x_{i_1} x_{i_2} \ldots x_{i_n}$$

, show that PCP is undecidable by reduction from the halting problem.

Hint: Represent run of the TM using w_i and x_i, relying on the fact that TM computation is local

[1] You can find out more about Busy Beavers here and here.