COL352 Problem Sheet 4

Topics: Pumping Lemma for Context-Free Languages and Normal Forms

For the following problems you might find it useful to use the fact that the the intersection of a CFL and a regular language is also a CFL.

Problem 1. (Easy) Let

$$L = \{xyz \mid x, z \in \Sigma^*,\ y \in \Sigma^*1\Sigma^*1\Sigma^*,\ |x| = |z| \geq |y|\}.$$

Show that L is not a CFL.

Problem 2. (Medium) Let

$$CUT(A) = \{yxz \mid xyz \in A\}$$

Show that the class of CFLs is not closed under CUT.

Hint: CFLs are closed under intersection with regular languages.

Problem 3. (Easy) Show that if a grammar G is in Chomsky normal form then for any $x \in \Sigma^n, x \in L(G)$ for $n \geq 1$ then there are exactly 2n-1 steps in any derivation of x. (Recall a grammar is in chomsky normal form if all its rules are of the form $A \to c$ or $A \to BC$ where $c \in T$, $A, B, C \in NT$)

Problem 4. (Easy) Prove the below languages are not CFLs.

(a) **(Easy)** $\{0^n1^n0^n1^n \mid n \geq 0\}$

(b) **(Easy)** $\{w\#t \mid w \text{ is a substring of } t, \text{where } w, t \in \{a, b\}^*\}$

(c) **(Easy)** $\{t_1\#t_2\#...\#t_k \mid k \geq 2, t_i = t_j \text{ for some } i \neq j,\ t_i \in \{a, b\}^*\}$

Problem 5. (Easy) Let B be the language of all palindromes over $\{0, 1\}$ containing equal number of 0's and 1's. Show that B is not a CFL.

Problem 6. (Easy) Let

$$F = \{a^ib^j \mid \exists k \in \mathbb{N}, i = k * j\}$$

Show that F is not context free.

Problem 7. (Easy) Let G be a CFG in Chomsky normal form that contains b variables. Show that if G generates some string with a derivation having atleast 2^b steps, $L(G)$ is infinite.

Problem 8. (Medium) The perfect shuffle operation over two languages defined as

$$PERFECT\_SHUFFLE(A, B) = \{a_1 b_1 ... a_k b_k \mid a_i, b_i \in \Sigma, a_1 a_2 .. a_k \in A, b_1 b_2 .. b_k \in B\}$$

Show that CFLs are not closed under perfect shuffle.

Problem 9. (Hard) Consider

$$NOPREFIX(A) = \{w \mid w \in A, \text{ no proper prefix of } w \in A\}$$

$$NOEXTEND(A) = \{w \mid w \in A, \ w \text{ is not proper prefix of any string in } A\}$$

Show that CFLs are not closed under $NOPREFIX$ and $NOEXTEND$ operation

Hint: Use a language like $a^i b^j c^k$ with some conditions on i, j, k such that it reduces to $a^n b^n c^n$

Problem 10. (Medium) Let

$$Y = \{w \mid w = t_1 \# t_2 \# ... t_k \text{ for } k \geq 0, \text{each } t_i \in 1^*, \text{and } t_i \neq t_j, \text{whenever } i \neq j\}$$

, where $\Sigma = \{1, \#\}$. Prove that Y is not context free.

Hint: Case analysis on the string $\#1\#...\#1^p$

Problem 11. (Easy) Let,

$$SCRAMBLE(A) = \{w \mid \exists x \in A, x \text{ is a permutation of } w\}$$

Consider a regular language defined over a alphabet Σ with $|\Sigma| > 2$, show that $SCRAMBLE(A)$ is not a CFL.

Hint: Consider intersection with a regular lang. to construct a lang. to resemble $0^n 1^n 0^n$

Problem 12. (Medium) Let $A = \{wtw^R \mid w, t \in \{0, 1\}^* \text{ and } |w| = |t|\}$, prove A is not a CFL.

Hint: Consider intersection with a regular lang. to construct a lang. to resemble $0^n 1^n 0^n$

Problem 13. (Medium)

Show that the following languages are *not* context-free.

(a) $L_1 = \{a^m b^n \mid m * n \text{ is a perfect square}\}$.

(b) $L_2 = \{x \in \{0, 1\}^* \mid x \text{ is the binary representation of } 3^{n^2} \text{ for some } n \in \mathbb{N} \cup \{0\}\}$.

(c) $L_3 = \{x \in \{a, b\}^* \mid \#a(x) \text{ is a multiple of } \#b(x)\}$.

(d) $L_4 = \{xyx \mid x, y \in \{0, 1\}^*, \ |x| > 0, \ |y| > 0\}$.