COL352 Problem Sheet 3

February 14, 2025

Topics: Context-Free Grammars and Pushdown Automata

Problem 1. (Easy)

Let

$$L = \{w\#x \mid w^R \text{ is a contiguous substring of } x,\ w, x \in \{0,1\}^*\}.$$

Construct a CFG for L.

Problem 2. (Easy)

Let

$$L = \{x_1\#x_2\#\cdots\#x_k \mid k \geq 1,\ x_i \in \{a,b\}^*, \text{ and for some } i, j,\ x_i = x_j^R\}.$$

Construct a CFG for L.

Problem 3. (Medium)

Let

$$L = \{a^i b^j \mid i \neq j,\ i \neq 2j\}.$$

Construct a CFG for L.

Hint: Break the language into separate cases depending on the relationship between i and j.

Problem 4. (Medium)

For a string w, define

$$SCRAMBLE(w) = \{t \mid t \text{ is a permutation of } w\}.$$

For a language L, define

$$SCRAMBLE(L) = \{t \mid \exists w \in L \text{ such that } t \in SCRAMBLE(w)\}.$$

Let L be a regular language over $\{0,1\}$. Show that $SCRAMBLE(L)$ is context-free.

Hint: Think in terms of counting the number of 0s and 1s using the pumping property of regular languages.

Problem 5. (Easy)

Given the grammar

$$S \to SS \mid T, \qquad T \to aTb \mid ab,$$

show that it is ambiguous. Construct an equivalent unambiguous grammar.

Problem 6. (Easy)

Let

$$B = \{uv \mid u \in \Sigma^*,\ v \in \Sigma^* 1 \Sigma^*,\ |u| \geq |v|\}.$$

Construct both a PDA and a CFG that recognize B.

Problem 7. (Medium)

Let

$$L = \{xyz \mid x, z \in \Sigma^*,\ y \in \Sigma^* 1 \Sigma^*,\ |x| = |z| \geq |y|\}.$$

Show that L is context-free.

Hint: Use the stack to match x and z while ensuring y contains at least one 1*.*

Problem 8. (Medium)

For a language A, define the rotational closure

$$RC(A) = \{yx \mid xy \in A\}.$$

Show that the class of context-free languages is closed under rotational closure.

Hint: Use closure under substitution and homomorphism, or simulate splitting using nondeterminism.

Problem 9. (Medium)

Let $G = (V, \Sigma, R, S)$ be a grammar. Prove that for every $x \in L(G)$, there exists a parse tree with root S, yield x, and height at most

$$|V|(|x| + 1).$$

Hint: Use the pigeonhole principle on repeated variables along root-to-leaf paths.

Problem 10. (Medium)

An NPDA is called a *binary-stack NPDA* if its stack alphabet has size 2. Show that binary-stack NPDAs and general NPDAs recognise the same set of languages.

Hint: Encode each stack symbol as a fixed-length binary string.