COL352 Problem Sheet 2

January 16, 2025

Topics: Pumping Lemma, Myhill–Nerode Theorem, DFA Minimization, CFGs, CFLs

Problem 1. (Easy) Let $\Sigma = \{0, 1\}$.

(a) Let $A = \{0^k u 0^k \mid k \geq 1 \text{ and } u \in \Sigma^*\}$. Show that A is regular.

(b) Let $B = \{0^k 1 u 0^k \mid k \geq 1 \text{ and } u \in \Sigma^*\}$. Show that B is not regular.

Problem 2. (Easy) Let $\Sigma = \{0, 1, \#\}$ and define

$$C = \{x\#x^R \mid x \in \{0, 1\}^*\}.$$

Show that C is a context-free language.

Problem 3.(Medium) Let L be a CFL and R a regular language. Is $L \setminus R$ necessarily context-free? If not, give a counterexample; otherwise, give a proof.

Hint:Construct NPDA for this using L's NPDA

Problem 4.(Medium) For a symbol $a \in \Sigma$ and a language $L \subseteq \Sigma^*$, define

$$a^{-1}L = \{w \in \Sigma^* \mid aw \in L\}.$$

Show that if L is context-free, then $a^{-1}L$ is also context-free.

Hint:Construct NPDA for this using L's NPDA

Problem 5 (Efficiency of NFA) (Easy). For $k \geq 1$, define

$$L_k = \{x \in \{0, 1\}^* \mid |x| \geq k \text{ and the } k\text{th symbol of } x \text{ from the right is } 1\}.$$

(a) Prove that every DFA recognizing L_k has at least 2^k states.

(b) Show that there exists an NFA with $k + 1$ states that recognizes L_k.

Problem 6.(Easy) A *coNFA* is defined similarly to an NFA, except that it accepts an input string w if and only if *every* possible state it could be in after reading w is an accepting state. (By contrast, an NFA accepts w if there exists at least one accepting state it could be in after reading w.)
Show that the class of languages recognized by coNFAs is exactly the class of regular languages.

Problem 7.(Easy) Construct the minimal deterministic finite automaton D that recognizes the language

$L = \{x \in \{0,1\}^* \mid x$ is the binary representation (without leading zeros) of a number coprime with $6\}$.

Prove that D is minimal by exhibiting, for each pair of distinct states q, q' of D, a string $z_{q,q'} \in \{0,1\}^*$ such that exactly one of

$$\delta^*(q, z_{q,q'}) \quad \text{and} \quad \delta^*(q', z_{q,q'})$$

is an accepting state of D.

Problem 8.(Medium) Prove that for any infinite regular language L, there exist two infinite regular languages L_1 and L_2 such that

$$L = L_1 \cup L_2 \quad \text{and} \quad L_1 \cap L_2 = \varnothing.$$

Hint: Use pumping lemma to construct these sets

Problem 9. Prove that the following languages are not regular.

(a) **(Easy)**$\{xx \mid x \in \{0,1\}^*\}$

(b) **(Easy)**$\{x \in \{0,1\}^* \mid x = x^R\}$ (palindromes)

(c) **(Medium)** $\{0^{n_1}1\,0^{n_2}1 \cdots 0^{n_k}1 \mid k \geq 1,\ n_1, n_2, \ldots, n_k \in \mathbb{N},\ n_1, \ldots, n_k \text{ are distinct}\}$

(d) **(Medium)**$\{xyx \mid x, y \in \{0,1\}^*,\ |x| > 0,\ |y| > 0\}$

(e) **(Hard)** $\{x \in \{0,1\}^* \mid x$ is the binary representation (without leading zeros) of 3^{n^2} for some $n \in \mathbb{N}\}$

(f) **(Hard)** $\{x \in \{0,1\}^* \mid x$ is the binary representation (without leading zeros) of $n!$ for some $n \in \mathbb{N}\}$

Hint: for e,f considering using length of strings in the language

Problem 10.(Easy) Design a context-free grammar that generates the language

$$\{x \in \{0,1\}^* \mid \#_0(x) = \#_1(x)\},$$

where $\#_0(x)$ and $\#_1(x)$ denote the number of occurrences of 0 and 1 in x, respectively.

Problem 11.(Easy) Let M_1 and M_2 be deterministic finite automata with k_1 and k_2 states, respectively, and let $U = L(M_1) \cup L(M_2)$.

(a) Show that if $U \neq \varnothing$, then U contains some string s such that $|s| < \max(k_1, k_2)$.

(b) Show that if $U \neq \Sigma^*$, then U excludes some string s such that $|s| < k_1 k_2$.

Problem 12. (Easy) For each of the following languages, determine whether they are regular by using the Myhill–Nerode theorem. If the language is regular, construct the minimal DFA that recognizes it. If it is not regular, prove it.

(a) $L_1 = \{w \in \{0,1\}^* \mid w \text{ ends with } 00\}$

(b) $L_2 = \{w \in \{0,1\}^* \mid w \text{ contains the substring } 010\}$

(c) $L_3 = \{w \in \{0,1\}^* \mid w \text{ has an even number of 0s and an even number of 1s}\}$

(d) $L_4 = \{a^n b^m \mid n \geq m\}$

(e) $L_5 = \{0^m 1^n \mid m \neq n\}$

(f) $L_6 = \{x \in \{a,b,c\}^* \mid x \text{ contains an equal number of occurrences of } ab \text{ and } ba \text{ as substrings}\}$

Problem 13.(Medium) The *rotational closure* of a language A is defined as

$$\mathrm{RC}(A) = \{yx \mid xy \in A\}.$$

(a) Show that for any language A, we have $\mathrm{RC}(A) = \mathrm{RC}(\mathrm{RC}(A))$.

(b) Show that the class of regular languages is closed under rotational closure.

Hint: For b construct an NFA which guesses what $\hat{\delta}(q_0, x)$ for A's DFA would be

Problem 14.(Hard) Let A be any language. Define

$$A_{\frac{1}{3}-\frac{1}{3}} = \{xz \mid \text{there exists } y \text{ such that } |x| = |y| = |z| \text{ and } xyz \in A\}.$$

In other words, $A_{\frac{1}{3}-\frac{1}{3}}$ is the set of all strings obtained from strings in A by removing their middle third.
Show that if A is regular, then $A_{\frac{1}{3}-\frac{1}{3}}$ is not necessarily regular.
Hint: Choose suitable A so intersection of $A_{\frac{1}{3}-\frac{1}{3}}$ with 0^*1^* is clearly not regular

Problem 15. (Easy) Show that the following languages are not regular.

(a) $\{0^n 1^n 0^n \mid n \geq 0\}$

(b) $\{w \in \{0,1\}^* \mid \#_0(w) = (\#_1(w))^2\}$

(c) $\{xy \mid x, y \in \{0,1\}^*,\ |x| = |y|,\ x \neq y\}$

(d) $\{ww^R w \mid w \in \{0,1\}^*\}$

(e) $\{0^p \mid p \text{ is prime}\}$

(f) $\{w \in \{a,b\}^* \mid w \text{ has twice as many } a\text{s as } b\text{s}\}$

Problem 16.(Easy) For each of the following languages, determine whether it is regular. If it is regular, construct a DFA recognizing it; if it is not regular, prove so.

(a) $\{0^i 1^j \mid i, j \geq 0 \text{ and } i \mid j\}$

(b) $\{x \in \{0,1\}^* \mid x = uvu \text{ for some } u, v\}$

Problem 17.(Hard) Let r_1, r_2 be rational numbers with $r_1 \leq r_2$. Define

$$L = \{a^i b^j \mid r_1 \leq \frac{i}{j} \leq r_2\}.$$

Is L a CFL? What happens if r_1, r_2 are arbitrary real numbers?

Construct CFG where $\frac{i}{j} = r_1, r_2$ can try to "combine" these grammars.

Problem 18.(Hard) Let $L = \{x\#y| \quad x, y \in \{0,1\}^*, x \neq y\}$. Show that L is a context-free language.

Hint: Construct NPDA which checks if $x[i] \neq y[i]$ for some i, extend it to match L

Problem 19.(Hard) Let $L = \{xy| \quad x, y \in \{0,1\}^*, |x| = |y|, x \neq y\}$. Show that L is a context-free language.

Hint: Construct CFG which ensures that i-th character of $x[i] \neq y[i]$ for some i

Problem 20. (Medium) Give *unambiguous* context–free grammars (CFGs) for the following languages over the alphabet $\{a, b\}$:

1. $\{\, w \mid \text{in every prefix of } w,\ \#a \geq \#b \,\}$
2. $\{\, w \mid \#a(w) = \#b(w) \,\}$
3. $\{\, w \mid \#a(w) \geq \#b(w) \,\}$

Hint: Try finding some pattern in these Languages.