COL352 Problem Sheet 1

January 16, 2025

Problem 1. (Easy) Construct a DFA over the alphabet $\{0, 1\}$ that accepts the set of all binary representations of natural numbers n such that $n \bmod 6 \in \{2, 5\}$.

Problem 2. (Medium) Construct a DFA over the alphabet $\{0, 1\}$ that accepts all strings in which the number of occurrences of the substring 01 is equal to the number of occurrences of the substring 10.

Hint: The key insight is that these counts differ by at most 1 in any string.

Problem 3. (Medium) Let

$$C_n = \{x \mid x \text{ is the binary representation of a number divisible by } n\}.$$

Prove that C_n is regular for all $n \geq 1$ and give a general DFA construction for C_n.

Hint: DFA approach: only n possible remainders

Problem 4. (Medium-Hard) Given DFAs D_1 and D_2 recognizing languages L_1 and L_2 respectively, determine whether the class of regular languages is closed under the following operations. Justify your answer by proof or counterexample.

1. Symmetric Difference: $L_1 \setminus L_2$

   Hint: Similar to union of regular languages

2. Perfect Shuffle: $A_1 = \{x_1y_1 \dots x_ny_n | x_1 \dots x_n \in L_1, y_1 \dots y_n \in L_2, x_i, y_i \in \Sigma\}$

   Hint: DFA that alternates between D_1 and D_2

3. k Perfect Shuffle: $A_k = \{x_1x_2...x_ky_1x_{k+1}x_{k+2} \dots x_{2k} \dots x_{nk}y_n | x_1 \dots x_{nk} \in L_1, y_1 \dots y_n \in L_2, x_i, y_i \in \Sigma\}$ i.e. k characters of a string in L_1 followed by one in L_2.

   Hint: Use DFA with counter modulo k

4. Shuffle: $A = \{x_1y_1 \dots x_ny_n | x_1 \dots x_n \in L_1, y_1 \dots y_n \in L_2, x_i, y_i \in \Sigma^*\}$
   (Note difference from subproblem 3: here, $x_i \in \Sigma^*$ not Σ)

   Hint: Use non-determinism to switch between D_1 and D_2.

Problem 5. (Medium) Design an algorithm that takes as input a DFA D and decides whether the language $L(D)$ is:

1. Empty
2. Σ^*

3. Finite
4. $\{0^{a+bn} : n \geq 0, a, b \in \mathrm{N}\}$, given $|\Sigma| = \{0\}$.

Hint: Think of it as a Graph Reachability problem

Problem 6. **(Medium)** Let L_1 be a regular language and L_2 be any language over the same alphabet. Prove that

$$L_1/L_2 = \{x \mid \exists y \in L_2 \text{ such that } xy \in L_1\}$$

is regular by explicitly constructing a DFA for L.

Hint: Modify accepting states of M_1.

Problem 7. **(Hard)** Let L be a regular language. Let x^R be the reverse of string x. Prove that the following languages are regular:

1. $\{x \mid x \cdot x^R \in L\}$
2. $\{x \mid x \cdot x^R \cdot x \in L\}$
3. $\{x \mid x^3 \in L\}$

Hint: 2 pointer approach, use non determinism to guess state of M_L after reading x before reading x^R.

Problem 8. **(Easy)** A function $h : \Sigma^* \to \Gamma^*$ is called a homomorphism if $h(xy) = h(x)h(y)$ for all $x, y \in \Sigma^*$. Prove that if L is a regular language, then $h(L)$ is also regular.

Problem 9. **(Easy)** Let

$$L_1 = (0|1)^*0(0|1)^*1(0|1)^*, \quad L_2 = (0|1)^*01(0|1)^*.$$

Show that $L_1 = L_2$.

Problem 10. **(Medium)** Give a regular expression over $\{0, 1\}$ that describes the language constructed in Problem 1.

Hint: Start from the DFA in Problem 1 and convert to regex

Problem 11. **(Medium)** Given two DFAs D_1 and D_2, design an efficient algorithm to determine whether

$$L(D_1) \subseteq L(D_2).$$

Problem 12. **(Medium)** Prove that the class of regular languages is closed under inverse homomorphisms.

Problem 13. **(Easy)** Let L be a regular language. Define

$$\mathrm{Pre}(L) = \{x \mid \exists y \text{ such that } xy \in L\}.$$

Prove that $\text{Pre}(L)$ is regular by constructing an automaton.

Problem 14. (Easy-Medium) Let L be a regular language. Define

$$\text{Suf}(L) = \{x \mid \exists y \text{ such that } yx \in L\}.$$

Prove that $\text{Suf}(L)$ is regular by constructing an automaton.

Problem 15. (Medium) Let A be a language. Define

$$\text{INSERT}(A) = \{xay \mid xy \in A,\ a \in \Sigma\}.$$

Show that the class of regular languages is closed under the INSERT operation.

Hint: Non-Determinism to Rescue.

Problem 16. (Medium) Let A be a language. Define

$$\text{DROPOUT}(A) = \{xz \mid xyz \in A,\ y \in \Sigma\}.$$

Show that the class of regular languages is closed under the DROPOUT operation.

Hint: Non-Determinism to Rescue.

Problem 17. (Hard) Let A be a language. Define

$$A_{1/2} = \{x \mid \exists y \quad xy \in A, |x| = |y|, y \in \Sigma^*\}.$$

Show that the class of regular languages is closed under the 1/2 operation. Similarly define the operation $A_{m/n}$. Comment on them too.

Hint: Use non determinism to guess y

Problem 18. (Medium) Let $A, B \subseteq \Sigma^*$ be languages. Define the *avoids* operation as

$$A \text{ avoids } B = \{\, w \mid w \in A \text{ and } w \text{ does not contain any string in } B \text{ as a substring} \,\}.$$

Prove that the class of regular languages is closed under the avoids operation.

Hint: Product construction with Non Determinism

Problem 19. (Medium) Let A $= (q, \Sigma, \delta, Q, F)$ be an NFA. Let the universal recognized language U(A) of A be defined as follows.

$$U(A) = \{w \in \Sigma^* \mid \hat{\delta}(Q, w) \subseteq F\}$$

1. Prove/Disprove $U(A) \subseteq L(A)$.
2. Prove/Disprove that universal recognized languages are regular.

Hint: Use subset construction.