COL352 Problem Sheet 2

January 31, 2025

The problems marked as ** are relatively harder than the other problems

Problem 1. *Given a language L over alphabet Σ, define the language $cdr(L) = \{y \in \Sigma^* \mid ay \in L$ for some $a \in \Sigma\}$. Show that the regular languages are closed under cdr()*

Problem 2. *Conclude that a regex without negation and star always generates a finite language.*

Problem 3 (Efficiency of NFA). *Let $L_k = \{x \in \{0,1\}^* \mid |x| \geq k$ and the k'th character of x from the end is a 1$\}$. Prove that every DFA that recognizes L_k has at least 2^k states. Also show that, on the other hand, there is an NFA with $k+1$ states that recognizes L_k.*

Problem 4. *A coNFA is like an NFA, except it accepts an input w if and only if every possible state it could end up in when reading w is an accept state. (By contrast, an NFA accepts w iff there exists an accept state it could end up in when reading w.) Show that the class of languages recognized by coNFAs is exactly the regular languages.*

Problem 5. *Construct the minimal DFA D that recognizes the language*

$$\{x \in \{0,1\}^* \mid x \text{ is the binary representation of a number coprime with } 6\}.$$

Prove its minimality by giving a string $z_{q,q'}$ for each pair of distinct states q, q' such that exactly one of $\delta(q, z_{q,q'})$ and $\delta(q', z_{q,q'})$ is an accepting state of D.

Problem 6. *Prove that for any infinite regular language L, there exist two infinite regular languages L_1, L_2 such that $L = L_1 \cup L_2$ and $L_1 \cap L_2 = \emptyset$**.*

Problem 7. *Construct a **minimal** DFA which accept the language $L = \{w \mid w \in \{a,b\}^*$ and $Na(w) \mod 3 = Nb(w) \mod 3\}$, where $Na(w)$ and $Nb(w)$ return the number of occurrences of a and b in w respectively.*

Problem 8. *Prove that the following languages are not regular.*

1. $\{xx \mid x \in \{0,1\}^*\}$
2. $\{x \in \{0,1\}^* \mid x = reverse(x)\}$
3. $\{0^{n_1}10^{n_2}1 \cdots 0^{n_k}1 \mid k, n_1, n_2, \cdots, n_k \in \mathbb{N} \cup \{0\}$ *and* $n_1, \ldots, n_k$ *are distinct*$\}$
4. $\{xyx \mid x, y \in \{0,1\}^*$ *and* $|x| > 0,\ |y| > 0\}$
5. $\{x \in \{0,1\}^* \mid x$ *is the binary representation of* 3^{n^2}, *without leading 0's, for some* $n \in \mathbb{N}\}$
6. $\{0^m1^n \mid m \neq n\}$ *(As a challenge, construct a clean proof using the pumping lemma only.)*
7. $\{x \in \{a,b,c\}^* \mid x$ *contains an equal number of occurrences of ab and ba as substrings*$\}$
8. $\{x \in \{0,1\}^* \mid x$ *is the binary representation of* $n!$, *without leading 0's, for some* $n \in \mathbb{N}\}$

Problem 9. *Design a context free grammar for the language* $\{x \in \{0,1\}^* \mid \#0\text{'s in } x = \#1\text{'s in } x\}^{**}$.

Problem 10. *If* A *is a set of natural numbers and* k *is a natural number greater than 1, let*

$$B_k(A) = \{w | w \text{ is the representation in base } k \text{ of some number in } A\}$$

Here, we do not allow leading $0s$ *in the representation of a number. For example,* $B2(\{3,5\}) = \{11, 101\}$ *and* $B_3(\{3,5\}) = \{10, 12\}$. *Give an example of a set* A *for which* $B_2(A)$ *is regular but* $B_3(A)$ *is not regular. Prove that your example works.*

Problem 11. *Consider languages* B *and* C,

1. $B = \{1^k y | \ y \in \{0,1\}^*$ *and* y *contains at least* k $1s$*, for* $k \geq 1\}$. *Show that* B *is a regular language.*
2. $C = \{1^k y | \ y \in \{0,1\}^*$ *and* y *contains at most* k $1s$*, for* $k \geq 1\}$. *Show that* C *is not a regular language.*