Recognizability and Decidability

Recall: A Turing machine is essentially an FSA with infinite tape

The tape head can move left or right

$M = (Q, \Sigma, \Gamma, \delta, s, t, r)$ is a deterministic TM

Start in s with the input on the tape, tape head at its first letter

What changes with each letter of the input?

Current state, tape contents, tape head position

These form a configuration

Let $u, v \in \Gamma^*$, and $a, b, c \in \Gamma$. Then,

$$uaqbv \xrightarrow[M]{1} uacq'v \quad \text{iff} \quad \delta(q,b) = (q', c, R), \quad \text{and}$$

$$uaqbv \xrightarrow[M]{1} uq'acv \quad \text{iff} \quad \delta(q,b) = (q', c, L)$$

u, v generally taken to be the two "halves" of the input around head, bookending $\sqcup$ symbols ignored.

What is the "language" of a Turing machine $M = (Q, \Sigma, \Gamma, \delta, s, t, r)$?

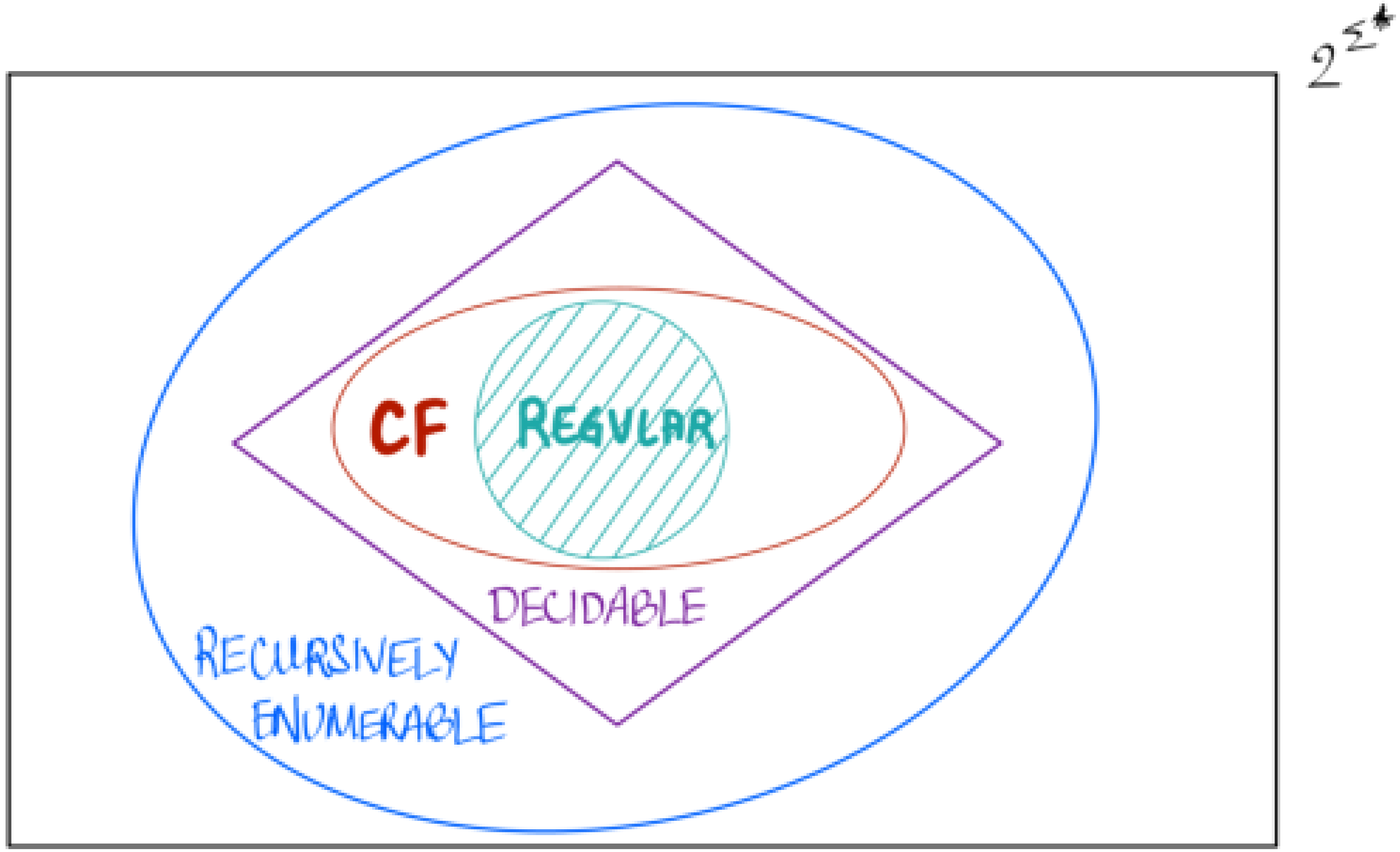
Consider $P_M = \{ w \mid \text{there are } u, v \in \Gamma^* \text{ s.t. } sw \xrightarrow[M]{*} utv \}$, and

$$N_M = \{ w \mid \text{there are } u, v \in \Gamma^* \text{ s.t. } sw \xrightarrow[M]{*} urv \}$$

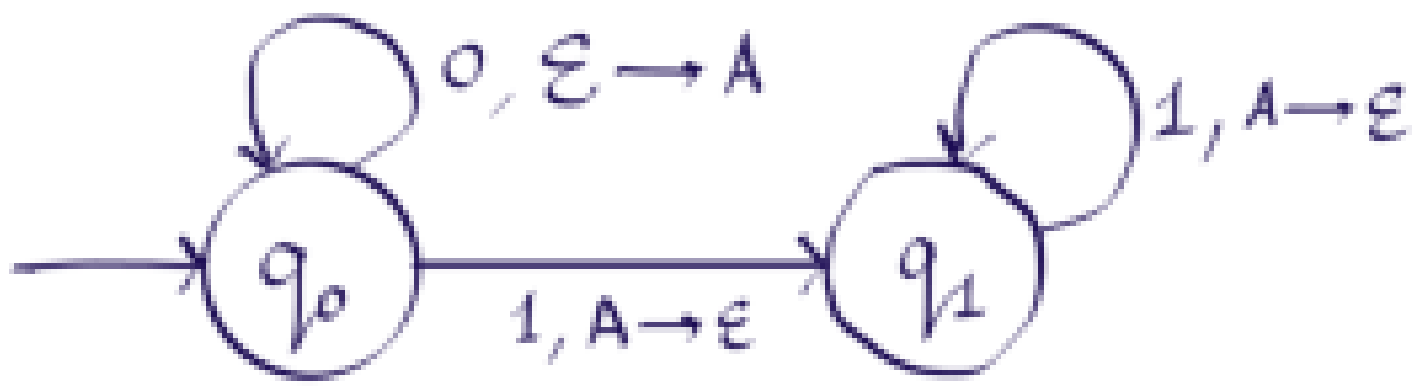
Suppose $\mathcal{L} = P_M$. Then, we say that $\mathcal{L}$ is recognized by M, and that $\mathcal{L}$ is Turing-recognizable, or recursively enumerable (r.e.)

If $\mathcal{L}$ is recognized by M, and in addition, $N_M = \{0,1\}^* \setminus P_M$, then we say that $\mathcal{L}$ is decided by M, and that $\mathcal{L}$ is decidable, or recursive

Decidability $\Rightarrow$ Turing-recognizability (but not the other way!)

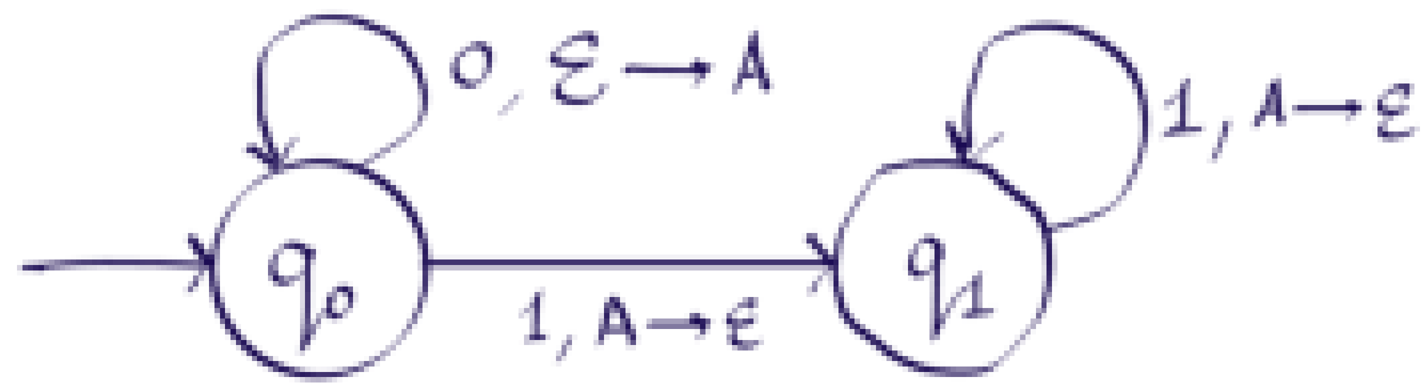

2^{Σ^*}
CF
REGULAR
DECIDABLE
RECURSIVELY ENUMERABLE

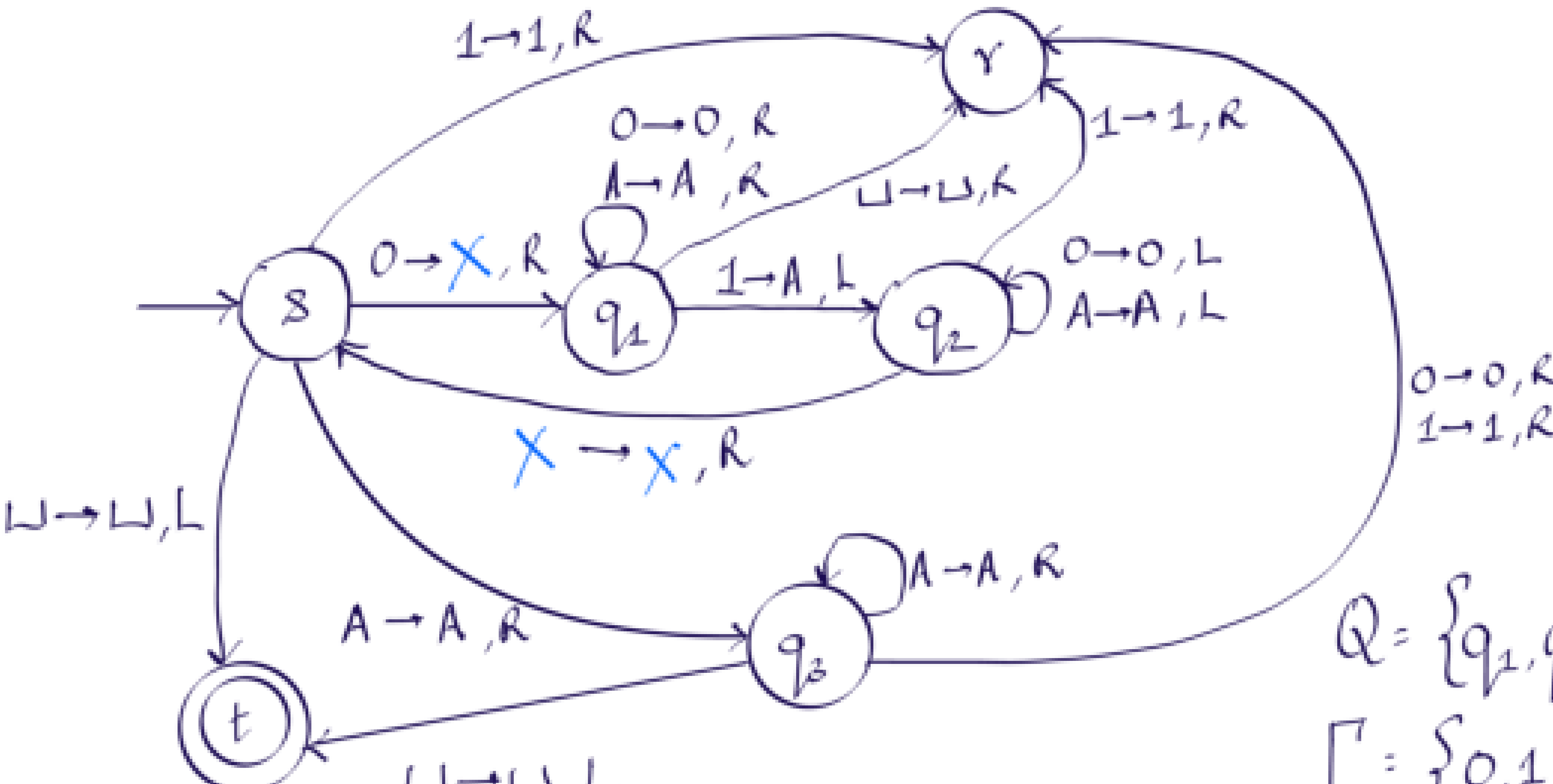
$$L = \{0^n 1^n \mid n \geq 0\}$$



$$M = (\{q_1, q_2, q_3\}, \{0, 1\}, \{0, 1, A\}, \delta, q_0, \phi)$$

$\mathcal{L} = \{ 0^n 1^n \mid n \geq 0 \}$





$Q = \{ q_1, q_2, q_3, s, t, r \}$

$\Gamma = \{ 0, 1, A, \sqcup, X \}$

$$\mathcal{L} = \{ \omega \# \omega \mid \omega \in \{0,1\}^* \} \subseteq \{0,1\}^* \# \{0,1\}^*$$

$\mathcal{L} = \{\omega\#\omega \mid \omega \in \{0,1\}^*\} \subseteq \{0,1\}^*\#\{0,1\}^*$

Start with the leftmost letter of ω, say 'c'

Replace it by A

Scroll right till a # is found*

If the symbol to the right of the # is 'c', replace it by A*

Scroll all the way back till the symbol to the left of the head is A

Repeat till the tape only has As followed by a # followed by As*

$$L = \{0^{2^n} \mid n \geq 0\}$$

$$\mathcal{L} = \{ 0^{2^n} \mid n \geq 0 \}$$

Start with the leftmost 0 of the word

Replace every second letter by A till you hit a blank

If only one 0, accept. If >1 odd 0s, reject.

Scroll all the way back till the symbol to the left of the head is A

Repeat till the tape has only As