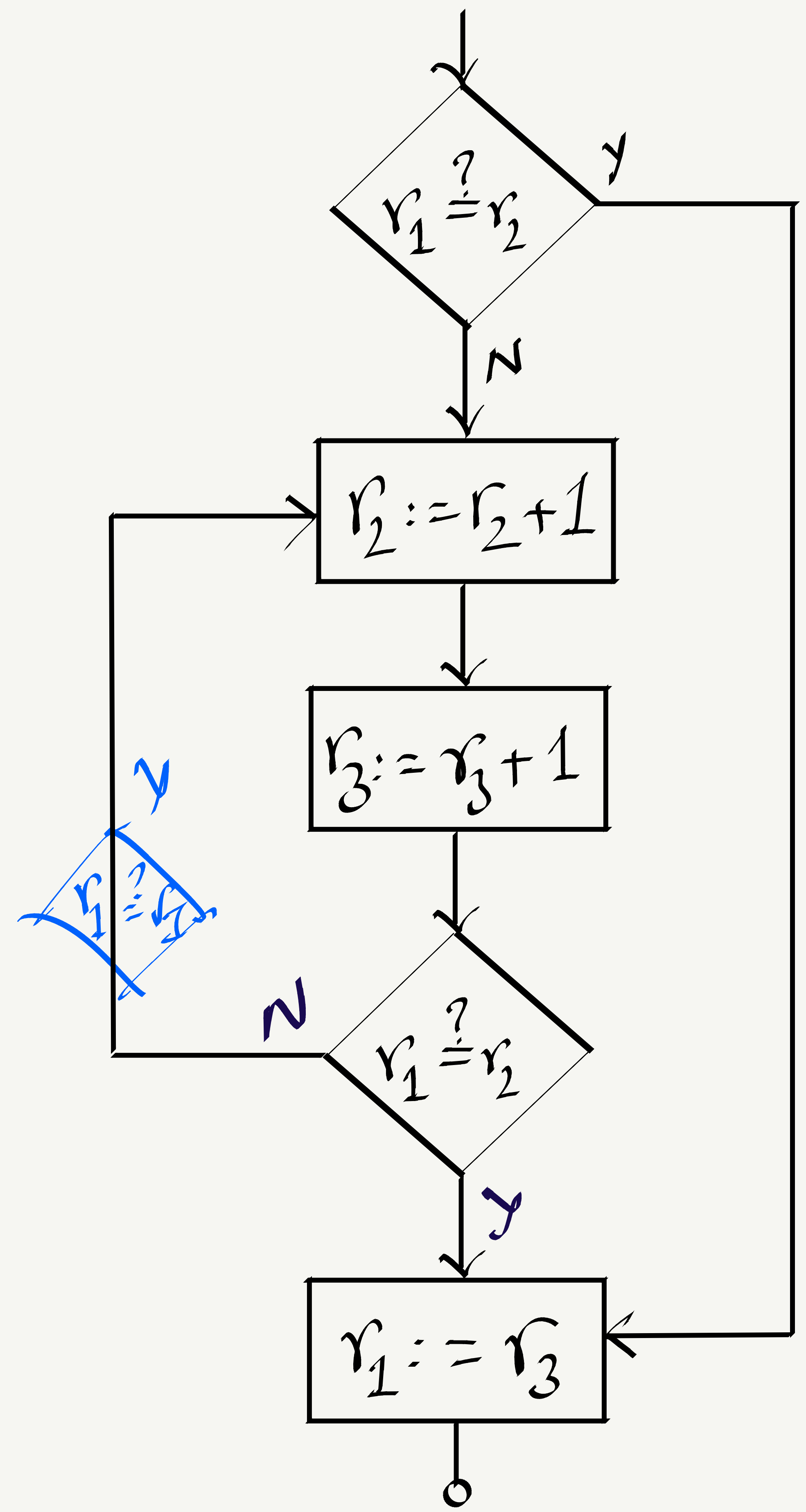
Computable Functions

Recall: Wanted to pin down a notion of "computable"
Isolated it to functions which have "efficient procedures"
Introduced a basic computer, the register machine (URM)

I_1: $J(1,2,6)$
I_2: $S(2)$
I_3: $S(3)$
I_4: $J(1,2,6)$
I_5: $J(1,1,2)$
I_6: $T(3,1)$



R_1	R_2	R_3
x	y	z
x	$y+1$	$z+1$
	$\vdots$	
x	$y+k$	$z+k$

$P(a_1, a_2, \ldots, a_n)$ denotes the program P with initial configuration
$a_1, a_2, \ldots, a_n, 0, 0, 0, \ldots\ldots$

$P(a_1, a_2, \ldots, a_n) \downarrow b$: P with initial config $a_1, \ldots, a_n$ terminates with $r_1 = b$.

For $f: \mathbb{N}^n \longrightarrow \mathbb{N}$, P URM-computes f if, for every $a_1, a_2, \ldots, a_n, b$,
$P(a_1, a_2, \ldots, a_n) \downarrow b$ if and only if
$(a_1, a_2, \ldots, a_n) \in \mathrm{Dom}(f)$, and $f(a_1, a_2, \ldots, a_n) = b$.

Any partial function $f: \mathbb{N}^n \longrightarrow \mathbb{N}$ is URM-computable if there is a program that URM-computes f.

Is $f(x,y) = x+y$ URM-computable?

$P(x,y,0)\downarrow(x+y)$

$I_1 : J(2,3,10)$

$I_2 : S(1)$

$I_3 : S(3)$

$I_4 : J(2,2,1)$

R_1	R_2	R_3
x	y	0
$x+1$	y	1
$x+2$	y	2
$\vdots$		
$x+k$	y	k
	$\vdots$	
$x+y$	y	y

Is $f(x,y) = x+y$ URM-computable?

$P(x,y,0)$ computes $f(x,y)$ where P is

$$I_1 : J(2,3,6)$$
$$I_2 : S(1)$$
$$I_3 : S(3)$$
$$I_4 : J(2,3,6)$$
$$I_5 : J(1,1,2)$$

$$\begin{array}{ccc} \boxed{x} & \boxed{y} & \boxed{0} \\ R_1 & R_2 & R_3 \end{array}$$

$$\begin{array}{ccc} \boxed{x+1} & \boxed{y} & \boxed{1} \\ R_1 & R_2 & R_3 \end{array}$$

$$\vdots$$

$$\begin{array}{ccc} \boxed{x+y} & \boxed{y} & \boxed{y} \\ R_1 & R_2 & R_3 \end{array}$$

$$f(x) = x \dot{-} 1 = \begin{cases} x-1, & x \geq 1 \\ 0, & x = 0 \end{cases}$$

$$f(x) = x \dot{-} 1 = \begin{cases} x-1, & x \geq 1 \\ 0, & x = 0 \end{cases}$$

$P(x, 0, 1, 0)$ computes $f(x)$, where P is as follows

$I_1 : J(1, 2, 10)$

$I_2 : J(1, 3, 6)$

$I_3 : S(4)$

$I_4 : S(3)$

$I_5 : J(1, 1, 2)$

$I_6 : T(4, 1)$

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while (r_1 ≠ r_3)
{
    r_4++;
    r_3++;
}
r_1 = r_4
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$\boxed{x}\ \boxed{0}\ \boxed{1}\ \boxed{0}$

$\boxed{x}\ \boxed{0}\ \boxed{2}\ \boxed{1}$

$\vdots$

$\boxed{x}\ \boxed{0}\ \boxed{x}\ \boxed{x \dot{-} 1}$

Some basic computable functions:

(1) Zero function : $Z(x) = 0$

(2) Successor function : $S(x) = x + 1$.

(3) Projection function: $U_i^n(x_1, \ldots, x_n) = x_i$

(1) P $\quad I_1 : Z(1) \longrightarrow r_1 = 0$

(2) $P(x)$ $\quad I_1 : S(1) \longrightarrow r_1{+}{+}$

(3) $P(x_1, \ldots, x_n)$ $\quad I_1 : T(i, 1)$

Consider the following function:

$$f(x,y) = \begin{cases} 1, & \text{if } x \text{ is a multiple of } y \\ 0, & \text{otherwise} \end{cases}$$

"x is a multiple of y" is a predicate which influences $f(x,y)$.

Given an n-ary predicate $M(x_1, \dots, x_n)$,
we say that M is decidable
if the following function is computable

$$f_M(x_1, \dots, x_n) = \begin{cases} 1, & \text{if } M(x_1, \dots, x_n) \text{ holds} \\ 0, & \text{otherwise} \end{cases}$$

M is said to be undecidable if f_M is not computable.

Is $M(x, y) := x \neq y$ decidable?

$$f_M(x, y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{otherwise} \end{cases}$$

$$P(x, y, 0) \downarrow (f_M(x, y))$$

$I_1 : J(1, 2, 3)$

$I_2 : S(3)$

$I_3 : T(3, 1)$

A URM only operates over natural numbers.

What if my function of interest was over, say, strings?

This is handled by means of a coding from any domain D to $\mathbb{N}$.

A coding is an injective $\alpha : D \to \mathbb{N}$

Now, a function $f : D \to D$ is coded by

$$f^* : \mathbb{N} \to \mathbb{N}, \text{ where}$$

$$f^* = \alpha \circ f \circ \alpha^{-1}$$

So we can say that f is computable if f^* is URM-computable

Consider the two functions

$$f(x) = x \dot{-} 1$$

$$g(x) = \begin{cases} 5, & \text{if } x = 0 \\ 29, & \text{otherwise} \end{cases}$$

Is $g \circ f$ computable?

Ideally, just concatenate programs for f and g

But what needs to be kept in mind?