NON-CONTEXT FREE LANGUAGES

CFLs are closed under union, concatenation, star....

What about intersection and negation?

Saw a few examples of languages for which we could not easily come up with CFGs/PDAs.

Common characteristics:

→ Needed to count beyond "linear"

→ Needed to "match" extremal ends of a string but after disturbing the stack in between

How does one prove that a language is not context-free?

The pumping lemma for CFLs.

Pumping lemma for CFLs :

Consider a language $\mathcal{L} \subseteq \Sigma^*$. If

$\forall$ for any $k \geq 0$,

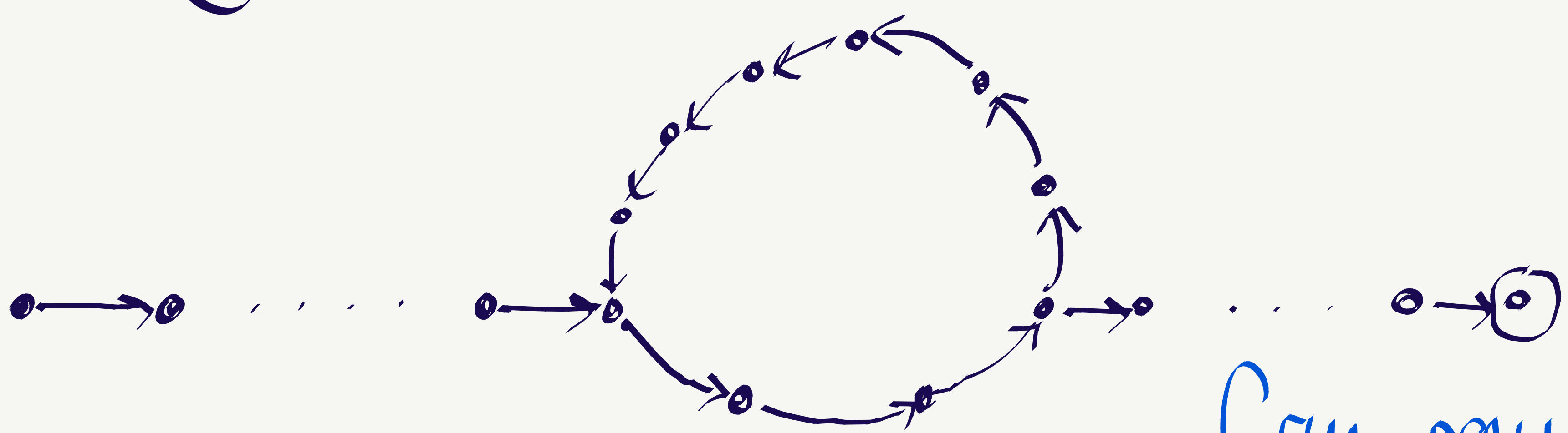
$\exists$ there is an $s \in \mathcal{L}$ s.t. $|s| \geq k$,

$\forall$ for any $x, u, v, w, z \in \Sigma^*$ s.t. $xuvwz = s$, $|uvw| \leq k$, $|uw| > 0$,

$\exists$ there is some $i \geq 0$ s.t. $xu^i v w^i z \notin \mathcal{L}$

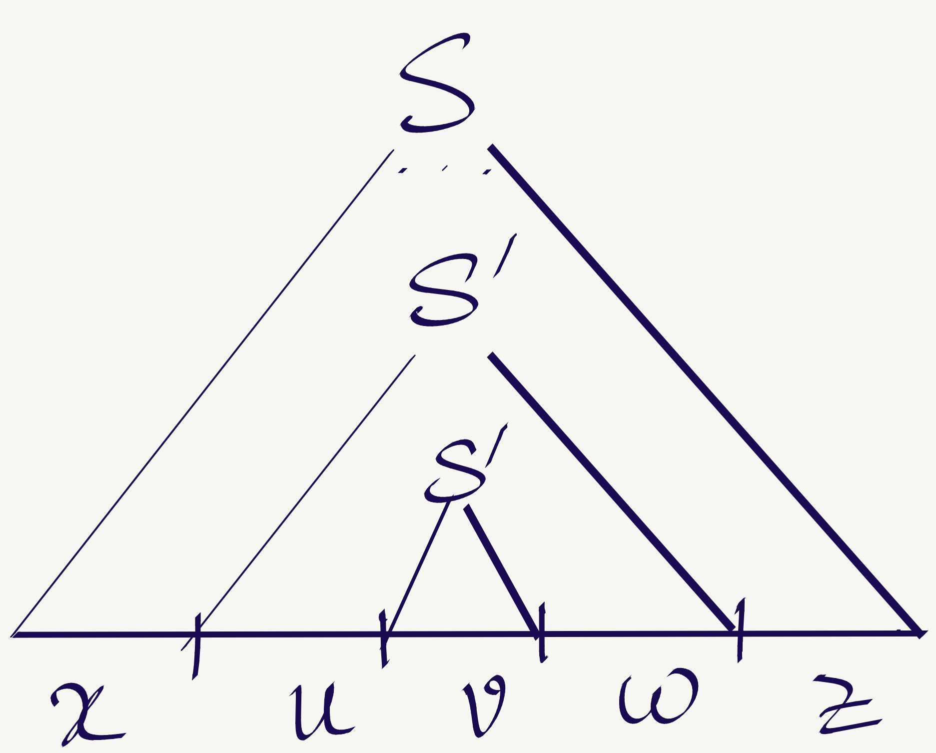
Then, $\mathcal{L}$ is not context-free.

For regular languages, we were basically exploiting the following structure of a DFA path



Can run the loop any number of times, and still stay in the language

For CFLs, we exploit the structure of parse trees



Cut and paste the section generated by the repeating non-terminal symbol any number of times, and still get a valid parse tree.

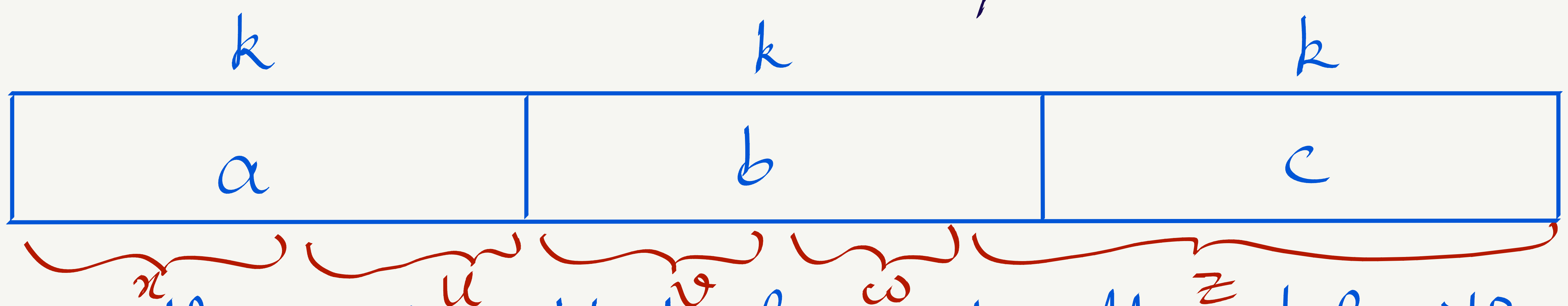
Use the pumping lemma to show that $\mathcal{L} = \{a^n b^n c^n \mid n \geq 0\}$ is not cf.

$\forall$ for any $k \geq 0$,

$\exists$ there is an $s \in \mathcal{L}$ s.t. $|s| \geq k$, $a^k b^k c^k$

$\forall$ for any $x, u, v, \omega, z \in \Sigma^*$ s.t. $xuv\omega z = s$, $|uv\omega| \leq k$, $|u\omega| > 0$,

$\exists$ there is some $i \geq 0$ s.t. $xu^i v \omega^i z \notin \mathcal{L}$ $i=2$



① $uv\omega$ falls exactly inside a block of size k: Mismatch with other two blocks

② $uv\omega$ spans blocks: Can only span two blocks, not three. Choosing $i=2$ will cause a mismatch with the third block.

So $\mathcal{L}$ is not context-free.

Use the pumping lemma to show that $\mathcal{L} = \{ww \mid w \in \{a,b\}^*\}$ is not CF.

$\forall$ for any $k \geq 0$,

$\exists$ there is an $s \in \mathcal{L}$ s.t. $|s| \geq k$, $a^k b^k a^k b^k$

$\forall$ for any $x, u, v, w, z \in \Sigma^*$ s.t. $xuvwz = s$, $|uvw| \leq k$, $|uw| > 0$,

$\exists$ there is some $i \geq 0$ s.t. $xu^i v w^i z \in \mathcal{L}$

Use the pumping lemma to show that $\mathcal{L} = \{ww \mid w \in \{a,b\}^*\}$ is not CF.

$\forall$ for any $k \geq 0$,

$\exists$ there is an $s \in \mathcal{L}$ s.t. $|s| \geq k$, $s = a^k b^k a^k b^k$

$\forall$ for any $x, u, v, w, z \in \Sigma^*$ s.t. $xuvwz = s$, $|uvw| \leq k$, $|uw| > 0$,

$\exists$ there is some $i \geq 0$ s.t. $xu^i v w^i z \notin \mathcal{L}$

k	k	k	k
a	b	a	b

Similar reasoning as earlier, set $i=2$.

Circle back to closure under the regular operations.

Are CFLs closed under intersection?

$\mathcal{L} = \{ a^n b^n c^n \mid n \geq 0 \}$ is not context-free

If we can express $\mathcal{L}$ as an intersection of two CFLs, that would prove non-closure under intersection.

$\{ a^m b^n c^n \mid m \geq 0, n \geq 0 \}$

$\{ a^n b^n c^m \mid m \geq 0, n \geq 0 \}$

Circle back to closure under the regular operations.

Are CFLs closed under intersection?

$$\mathcal{L} = \{ a^n b^n c^n \mid n \geq 0 \}$$ is not context-free

If we can express $\mathcal{L}$ as an intersection of two CFLs, that would prove non-closure under intersection.

$$\mathcal{L}_1 = \{ a^m b^n c^n \mid m, n \geq 0 \}$$

$$\mathcal{L}_2 = \{ a^n b^n c^m \mid m, n \geq 0 \}$$

* What about closure under negation?