CONTEXT-FREE

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PDA-RECOGNIZABLE

PART III

Recall: Given a PDA, construct a CFG generating the language of the PDA.

Modify the PDA so each transition either pushes or pops a symbol, and moves into a special accept state when it accepts with empty stack.

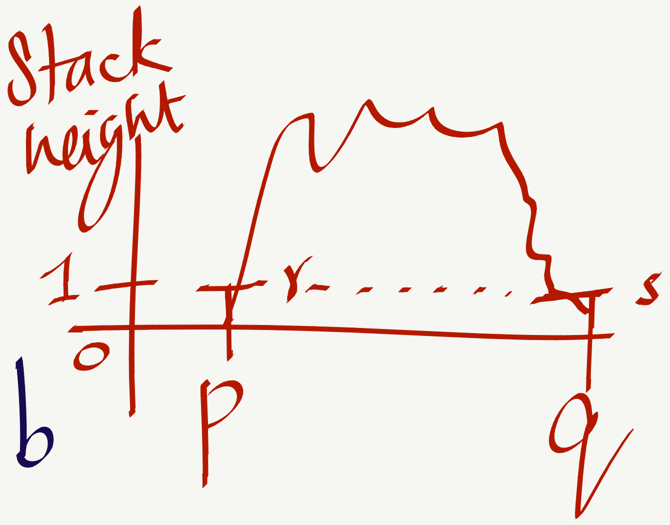
$$M = (Q, \Sigma, \Gamma, \Delta, q_0, \{f\})$$

$A_{pq} \in NT$: Generates all strings which take M from p with empty stack to q with empty stack

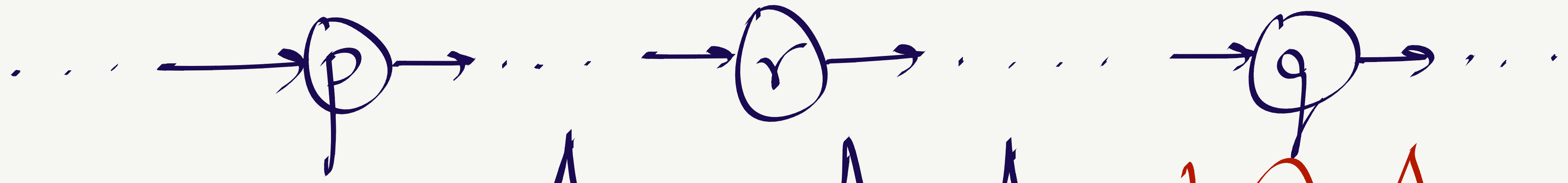
Inductive definition

① If stack is never empty except at p and q:
Symbol pushed on at p is popped off only just before q

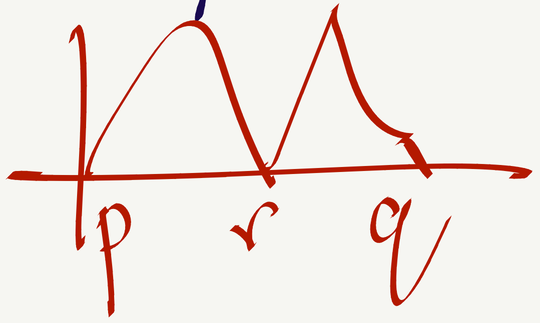
$A_{pq} \to aA_{rs}b$



② If stack is empty midway:



$A_{pq} \to A_{pr} A_{rq}$



Construction: Given a PDA $M = (Q, \Sigma, \Gamma, \Delta, q_0, \{f\})$, we construct $G = (NT, T, R, S)$ as follows:

$NT = \{A_{pq} \mid p, q \in Q\}$ $\qquad T = \Sigma$ $\qquad S = A_{q_0 f}$

$$R = \{A_{pp} \longrightarrow \varepsilon \mid p \in Q\} \cup \{A_{pq} \longrightarrow A_{pr} A_{rq} \mid p, q, r \in Q\} \cup$$

$$\left\{ A_{pq} \longrightarrow a A_{rs} b \;\middle|\; \begin{array}{l} p, q, r, s \in Q,\ a, b \in \Sigma \cup \{\varepsilon\},\ t \in \Gamma, \\ ((p, a, \varepsilon), (r, t)) \in \Delta, \\ ((s, b, t), (q, \varepsilon)) \in \Delta \end{array} \right\}$$

We now have to show our initial claim, that, for any $x \in \Sigma^*$, A_{pq} generates x iff x takes M from p with empty stack to q with empty stack. We prove each direction separately.

Claim: If A_{pq} generates x, then, x can take M from p with empty stack to q with empty stack.

Proof: By induction on n, the number of applications of rules from R in the derivation of x from A_{pq}.

$n = 1$: A derivation with only one rule applied which yields a string in Σ^* has only one candidate in R, $A_{pp} \to \varepsilon$. ε takes M from p to p without disturbing the stack.

$n = m+1$: Suppose A_{pq} generates x via $m+1$ applications of rules in R.

What is the first rule that can be applied?

Obviously not $A_{pp} \to \varepsilon$. Either $A_{pq} \to A_{pr} A_{rq}$ or $A_{pq} \to a A_{rs} b$.

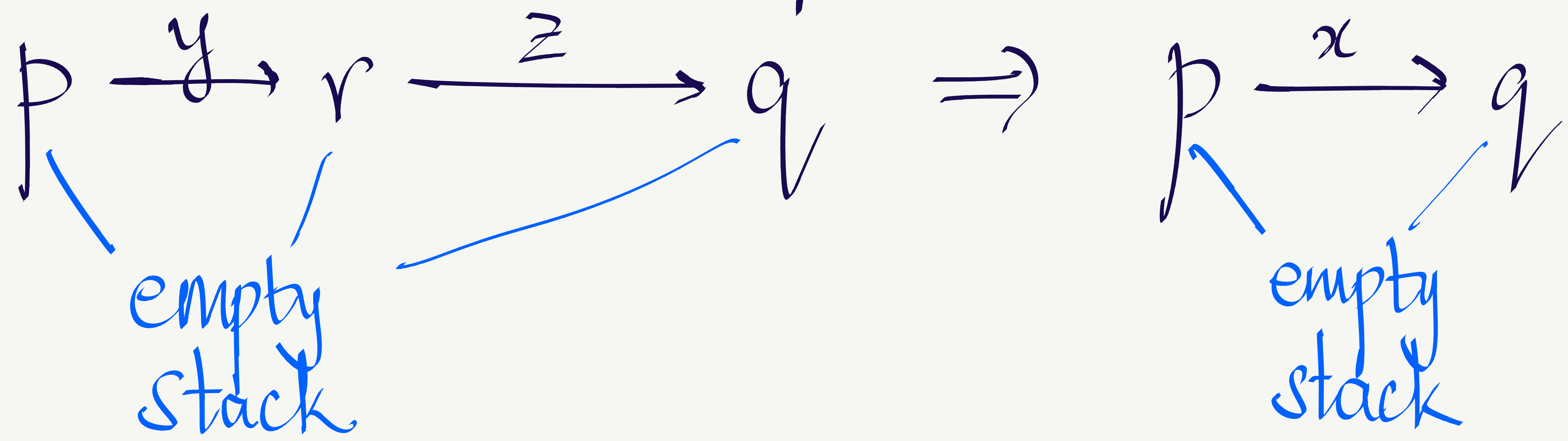
(a) Suppose the rule applied is of the form $A_{pq} \to A_{pr} A_{rq}$.

A_{pr} generates some $y \in \Sigma^*$, and A_{rq} generates some z s.t. $x = yz$.

By IH, y takes M from p to r and maintains empty stack at r

z takes M from r to q and maintains empty stack at q.

Suppose M is in state p, and the stack is empty.



(b) Suppose the rule applied is of the form $A_{pq} \to aA_{rs}b$.

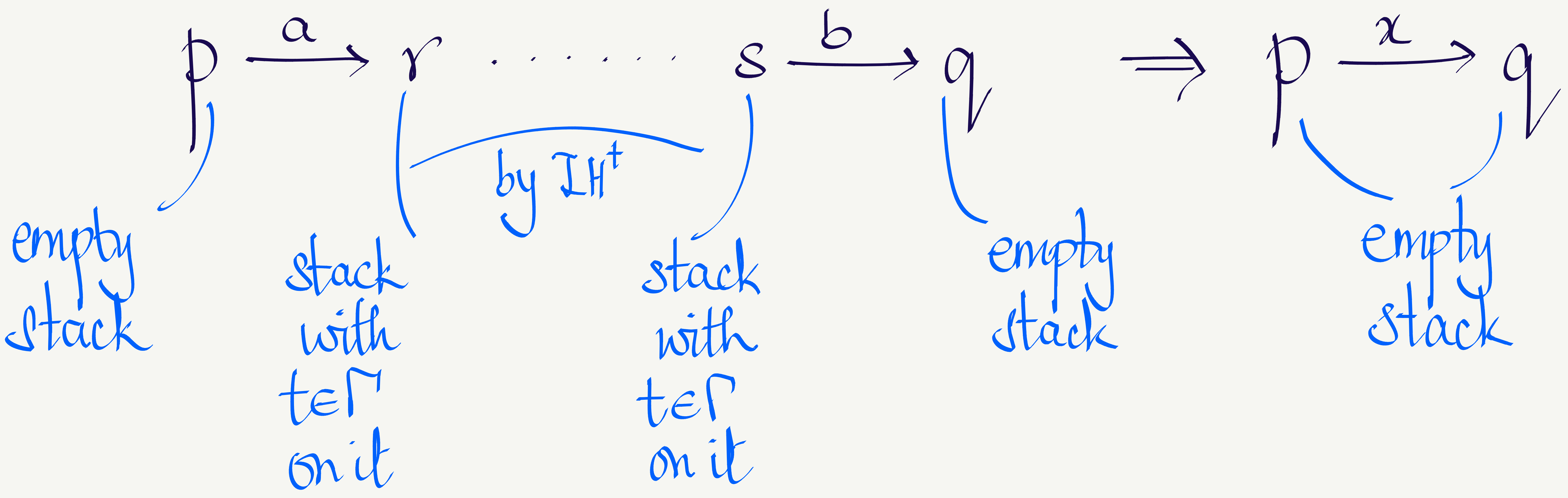
A_{rs} generates some y s.t. $x = ayb$.

By IH, y takes M from r to s with empty stack at both r and s.

Since $A_{pq} \to aA_{rs}b \in R$, we know that there is some $t \in \Gamma$ s.t.

$$\{((p,a,\varepsilon),(r,t)),\ ((s,b,t),(q,\varepsilon))\} \subseteq \Delta.$$

Suppose M is in state p.



□

Claim: If x takes M from p with empty stack to q with empty stack, then A_{pq} generates x by applications of rules in R.

Proof: There is some $n \geq 0$ s.t. $(p, x, \perp) \xrightarrow[M]{n} (q, \varepsilon, \perp)$.

Proof by induction on n.

$n = 0$: $(p, x, \perp) \xrightarrow[M]{0} (p, x, \perp) = (q, \varepsilon, \perp)$

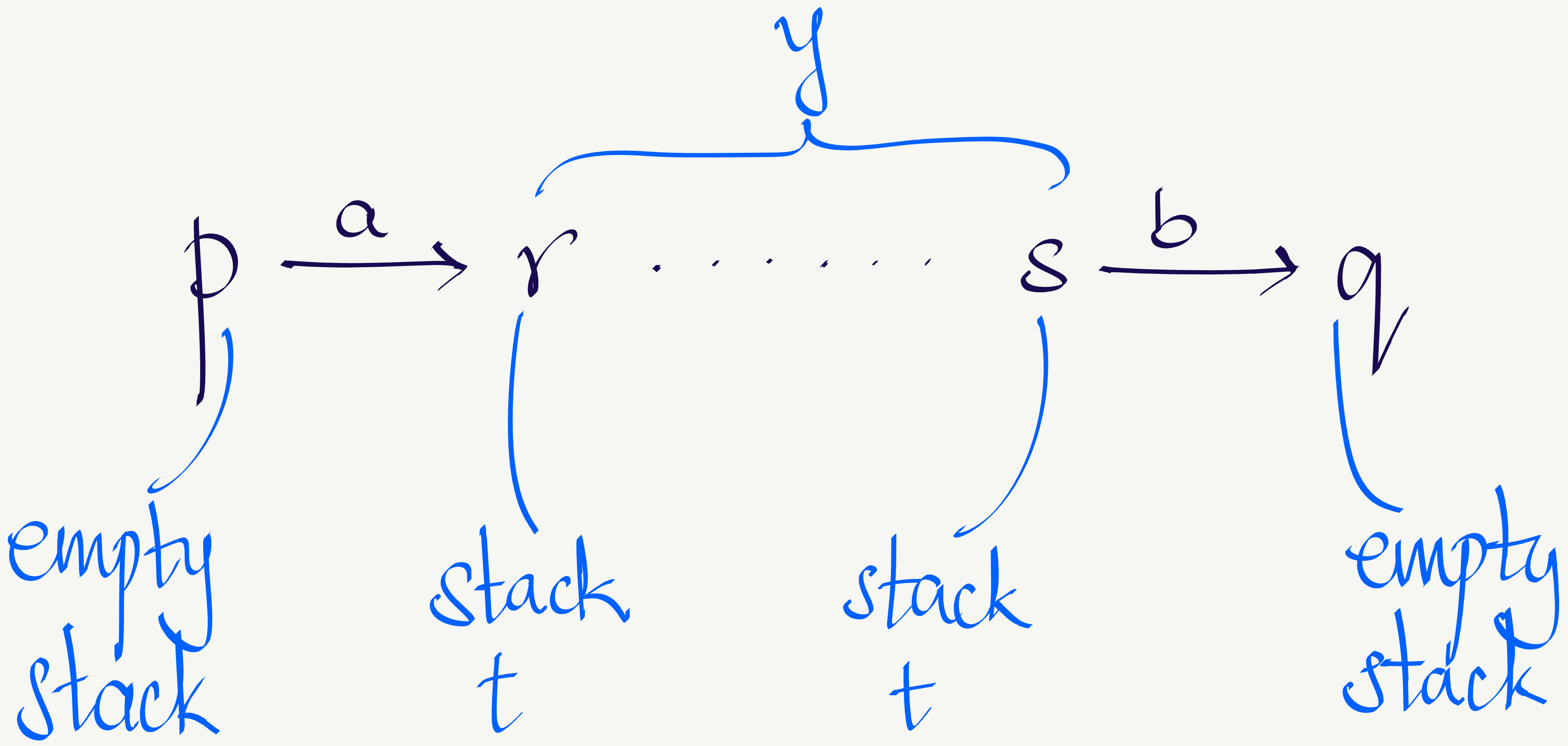
So $p = q$, and $x = \varepsilon$. $A_{pp} \to \varepsilon$, so done.

$n = m+1$: Two cases arise

(i) The stack is not empty anytime except at p and q:

The same symbol must be pushed and popped at the "ends"

Suppose we push a $t \in \Gamma$ to start with, and pop it off at the end.

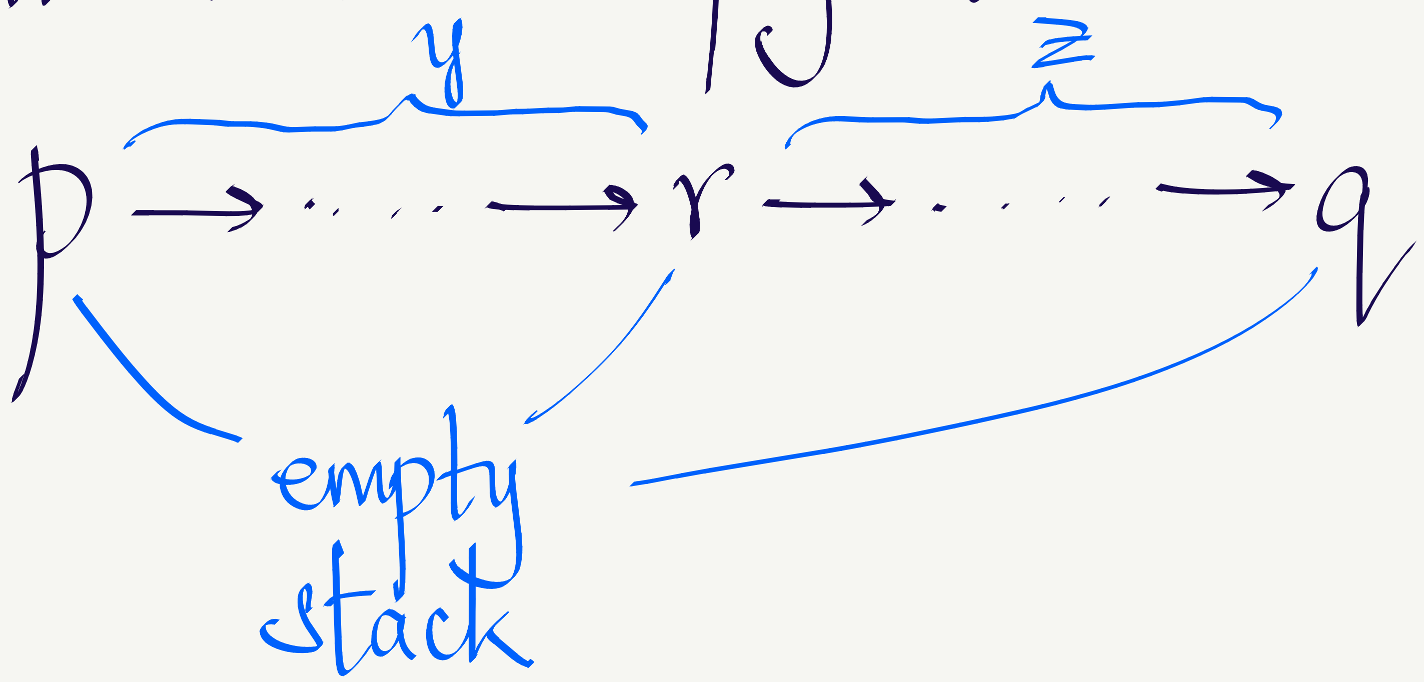


$x = ayb$, where

$A_{rs} \to y$ by IH

So $A_{pq} \to aA_{rs}b$.

(ii) The stack is empty at some midway point:



$x = yz$, where

$$\left.\begin{array}{l} A_{pr} \to y \\ A_{rq} \to z \end{array}\right\} \text{by IH}$$

So $A_{pq} \to A_{pr}A_{rq}$