Languages Accepted By PDAs

Recall: We formally defined pushdown automata as 6-tuples

$$M = (Q, \Sigma, \Gamma, \Delta, q_0, F)$$

Set of states

input alphabet

stack alphabet (contains $\perp$)

transition relation (nondeterministic!) $\Delta \subseteq \left(Q \times (\Sigma \cup \{\varepsilon\}) \times (\Gamma \cup \{\varepsilon\})\right) \times \left(Q \times \Gamma^*\right)$

initial state $\in Q$

accept states $\subseteq Q$.

A PDA which accepts $L = \{a^n b^n \mid n \geq 0\}$ by empty stack was

$$M = (\{q_0, q_1\}, \{a, b\}, \{\perp, c\}, \Delta, q_0, \phi), \text{ where}$$

$$\Delta = \{((q_0, a, \varepsilon), (q_0, c)), ((q_0, b, c), (q_1, \varepsilon)),$$
$$((q_1, b, c), (q_1, \varepsilon))\}$$

A configuration is an operational snapshot of the PDA, capturing its current state, remaining input string, and the current contents of the stack (read top to bottom).

Today: How do we show that $\mathcal{L}(M) = \mathcal{L} = \{a^n b^n \mid n \geq 0\}$?

① For every $w \in \mathcal{L}$, there is a sequence of moves which takes M from the initial configuration to a final configuration, and

② for every $w \in \{a, b\}^* \setminus \mathcal{L}$, no sequence of moves can take M from the initial configuration to a final configuration.

If we show this, we are done!

$$\Delta = \{((q_0, a, \varepsilon), (q_0, C)), ((q_0, b, C), (q_1, \varepsilon)), ((q_1, b, C), (q_1, \varepsilon))\}$$

Recall: Suppose $((q, a, A), (q', s)) \in \Delta$. Then,

$(q, aw, AS) \xrightarrow[M]{1} (q', w, sS)$, for any $w \in \Sigma^*$, $S \in \Gamma^*$.

What configuration changes are possible in one move of M?

$$(q_0, aw, S) \xrightarrow[M]{1} (q_0, w, CS)$$

$$(q_0, bw, CS) \xrightarrow[M]{1} (q_1, w, S)$$

$$(q_1, bw, CS) \xrightarrow[M]{1} (q_1, w, S)$$

$(q_0, ab, \perp) \rightarrow (q_0, b, C\perp)$
$\downarrow$
$(q_1, \varepsilon, \perp)$

$(q_0, aabb, \perp) \rightarrow (q_0, abb, C\perp)$
$\downarrow$
$(q_1, b, C\perp) \leftarrow (q_0, bb, CC\perp)$
$\downarrow$
$(q_1, \varepsilon, \perp)$

What does it mean for M to accept a string ω?

$$(q_0, \omega, \perp) \xrightarrow[M]{*} (q, \varepsilon, \perp) \quad \text{for some } q \in Q$$

$$\xrightarrow[M]{*} (f, \varepsilon, S) \quad \text{for } f \in F,\ S \in \Gamma^*.$$

Want to prove:

For any $\omega \in \Sigma^*$, $(q_0, \omega, \perp) \xrightarrow[M]{*} (q, \varepsilon, \perp)$ for some $q \in Q$ iff $\omega \in L$.

① $\omega = \varepsilon$: $(q_0, \varepsilon, \perp) \xrightarrow[M]{0} (q_0, \varepsilon, \perp)$

② $\omega = b\omega'$: $b\omega' \notin L$ for any ω', no transitions out of $(q_0, b\omega', \perp)$.

③ $\omega = a\omega'$: What do we do in this case?

To prove: For any $m, n, k \geq 0$, where $m+n>0$, there is $l>0$ s.t. $(q_0, a^m b^n, c^k \bot) \xrightarrow[M]{l} (q_1, \varepsilon, \bot)$ iff $n = m+k$.

Proof: By induction on m.

① $m=0$: Since $m+n>0$, $n \neq 0$. Two cases arise.

(a) $k=0$: There are no transitions out of $(q_0, b^n, \bot)$.

(b) $k>0$: $(q_0, b^n, c^k \bot) \xrightarrow[M]{1} (q_1, b^{n-1}, c^{k-1} \bot)$

Two cases arise.

(i) $n=1$: $(q_1, b^{n-1}, c^{k-1} \bot) = (q_1, \varepsilon, c^{k-1} \bot)$

There are no transitions out of $(q_1, \varepsilon, \_)$.

So $(q_0, b^1, c^k\bot) \xrightarrow[M]{*} (q_1, \varepsilon, \bot)$ iff $k=1$, i.e. $n = m+k$.

(ii) $n>1$: The only transition out of $(q_1, b^{n-1}, c^{k-1}\bot)$ is of the form $(q_1, b^{n-1}, c^{k-1}\bot) \xrightarrow[M]{1} (q_1, b^{n-2}, c^{k-2}\bot)$

Claim: For $\ell, p > 0$, $(q_1, b^\ell, c^p\bot) \xrightarrow[M]{*} (q_1, \varepsilon, \bot)$ iff $\ell = p$

Prove this by induction on ℓ.

So $(q_0, b^n, c^k\bot) \xrightarrow[M]{1} (q_1, b^{n-1}, c^{k-1}\bot) \xrightarrow[M]{*} (q_1, \varepsilon, \bot)$ iff $n-1 = k-1$ iff $n = k+0$ iff $n = k+m$.

This concludes the case when $m=0$.

② $m = \ell + 1$ for some $\ell \geq 0$:

Suppose $(q_0, a^{\ell+1}b^n, c^k\bot) \xrightarrow[M]{*} (q_1, \varepsilon, \bot)$.

$$(q_0, a^{\ell+1}b^n, c^k\bot) \xrightarrow[M]{1} (q_0, a^{\ell}b^n, c^{k+1}\bot)$$

this is the only possible move!

If $(q_0, a^{\ell}b^n, c^{k+1}\bot) \xrightarrow[M]{*} (q_1, \varepsilon, \bot)$, then by IH, $n = \ell + k + 1 = \ell + 1 + k = m + k$.

On the other hand, suppose $n = m + k = \ell + 1 + k$. Then,

$$(q_0, a^{\ell+1}b^{\ell+1+k}, c^k\bot) \xrightarrow[M]{1} (q_0, a^{\ell}b^{\ell+1+k}, c^{k+1}\bot) \xrightarrow[M]{*} (q_1, \varepsilon, \bot). \quad \square$$

by IH

Going back to our main proof,

③ $\omega = a\omega'$: Two cases arise

(a) $\omega = a^m b^n$ for some $m > 0$. Then,

$(q_0, \omega, \perp) = (q_0, a^m b^n, c^0 \perp)$. By the above,

$(q_0, a^m b^n, c^0 \perp) \xrightarrow[M]{*} (q_1, \varepsilon, \perp)$ iff $n = m + 0$ i.e. $n = m$.

(b) $\omega \neq a^m b^n$ for any m, n. Then, $\omega = a^m b^n \omega'$, where ω' starts with a.

Two cases arise.

Claim: $(q_0, a^m w, \perp) \xrightarrow[M]{m} (q_0, w, c^m \perp)$

Prove this by induction on m

$(q_0, b^n, c^k \perp) \xrightarrow[M]{*} (q_1, \varepsilon, \perp)$ iff $n = k$ (from earlier, since $m = 0$)

If $n > m$, putting these together,

$(q_0, a^m b^n w', \perp) \xrightarrow[M]{*} (q_1, b^{n-m} w', \perp)$ no transitions out of this configuration!

If $n \leq m$,

$(q_0, a^m b^n w', \perp) \xrightarrow[M]{m} (q_0, b^n w', c^m \perp) \xrightarrow[M]{*} (q_1, w', c^{m-n} \perp)$

w' starts with a, so no transitions out of this configuration!

Design a PDA which recognizes by empty stack the language

$$\mathcal{L}_{pal} = \{ \omega \cdot rev(\omega) \mid \omega \in \{a, b\}^* \}$$

What behaviour do we expect on a string abba? $abba \in \mathcal{L}$

$$(q_0, abba, \bot) \xrightarrow{\text{push}} (\_, bba, A\bot) \xrightarrow{\text{push}} (\_, ba, BA\bot) \xrightarrow{\text{pop}} (\_, a, A\bot) \xrightarrow{\text{pop}} (\_, \varepsilon, \bot)$$

guess that ω is over, and now we read $rev(\omega)$

What about on a string abb? $abb \notin \mathcal{L}$.

$$\Delta = \{((q_0, a, \varepsilon), (q_0, A)), ((q_0, b, \varepsilon), (q_0, B)),$$
$$((q_0, a, A), (q_1, \varepsilon)), ((q_0, b, B), (q_1, \varepsilon)),$$
$$((q_1, a, A), (q_1, \varepsilon)), ((q_1, b, B), (q_1, \varepsilon))\}$$

Claim: $M = (\{q_0, q_1\}, \{a, b\}, \{A, B, \perp\}, \Delta, q_0, \phi)$ accepts $\mathcal{L}_{pal}$ by empty stack.

What 1-move changes in configuration are possible?

$$(q_0, aw, S) \xrightarrow[M]{1} (q_0, w, AS)$$

$$(q_0, bw, S) \xrightarrow[M]{1} (q_0, w, BS)$$

$$(q_0, aw, AS) \xrightarrow[M]{1} (q_1, w, S)$$

$$(q_0, bw, BS) \xrightarrow[M]{1} (q_1, w, S)$$

$$(q_1, aw, AS) \xrightarrow[M]{1} (q_1, w, S)$$

$$(q_1, bw, BS) \xrightarrow[M]{1} (q_1, w, S)$$

For which $w \in \Sigma^*$ is it true that $(q_0, w, \perp) \xrightarrow[M]{*} (q, \varepsilon, \perp)$ for some $q \in Q$?

$w = \varepsilon$: trivially done

$w = aw'$:

$$(q_0, aw', \perp) \xrightarrow[M]{1} (q_0, w', A\perp)$$

$w = bw'$:

$$(q_0, bw', \perp) \xrightarrow[M]{1} (q_0, w', B\perp)$$

and then what?

Try a few examples, figure out a general pattern.

$$(q_0, abba, \perp) \xrightarrow[M]{*} (q_1, \varepsilon, \perp)$$

$$(q_0, ba, BA\perp) \xrightarrow[M]{*} (q_1, \varepsilon, \perp)$$

$$(q_0, bba, A\perp) \xrightarrow[M]{*} (q_1, \varepsilon, \perp)$$

$$(q_0, aabbaa, \perp) \xrightarrow[M]{*} (q_1, \varepsilon, \perp)$$

$$(q_0, abbaa, A\perp) \xrightarrow[M]{*} (q_1, \varepsilon, \perp)$$

$$(q_0, baba, \perp) \not\xrightarrow[M]{*} (q, \varepsilon, \perp) \text{ for any } q \in Q$$

$$(q_0, ba, AB\perp) \not\xrightarrow[M]{*} (q, \varepsilon, \perp) \text{ for any } q \in Q$$