Pushdown Automata

Recall: A context-free language is one which is generated by a CFG.

Examples: Strings with an equal number of 'a's and 'b's

Strings with balanced parentheses

Strings which are palindromes over $\Sigma = \{a, b\}$ etc.

Exercise: Construct an unambiguous CFG for this language

Today: A machine model for context-free languages

We said that regular expressions code up the class of languages recognized by NFAs/DFAs.

What is the equivalent machine model for context-free languages?

Consider an NFA with access to a global stack. One can

- push a symbol onto the stack
- pop the top symbol off the (non-empty) stack

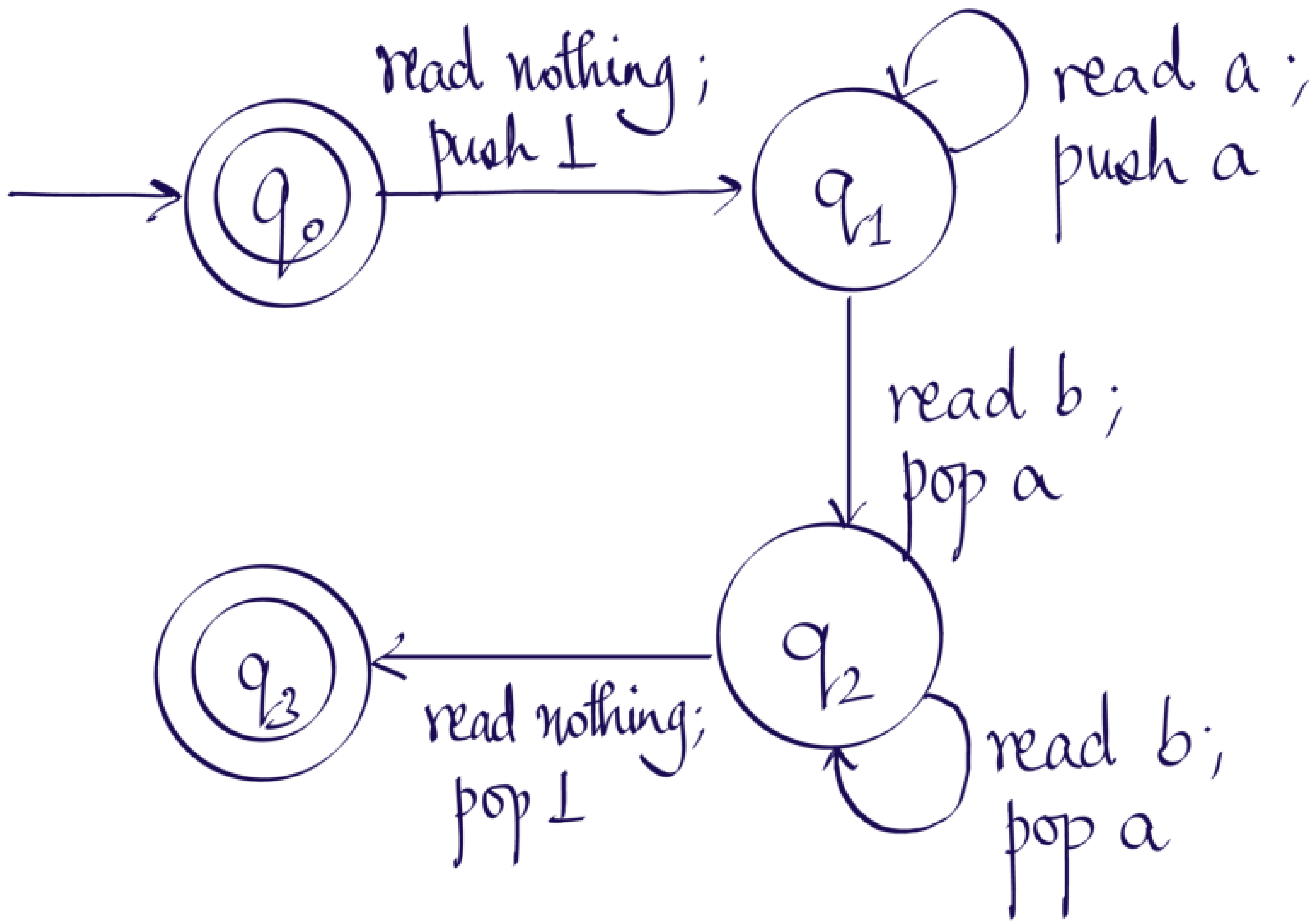
So how do we recognize $\mathcal{L} = \{a^n b^n \mid n \geq 0\}$?

Start in the initial state; stack contains an end marker $\perp$.

Consume input letters one at a time:

- if you see an 'a', push it onto the stack
- if you see a 'b', pop off the top symbol, as long as it is not $\perp$

When do we accept?



* pop operations get stuck if the top letter in the stack is NOT the letter required to be popped.

Pushdown automata (PDA): A 6-tuple $(Q, \Sigma, \Gamma, \Delta, q_0, F)$

Q: set of states $\qquad \Sigma$: input alphabet $\qquad \Gamma$: stack alphabet, $\perp \in \Gamma$.

$\Delta \subseteq \left(Q \times (\Sigma \cup \{\varepsilon\}) \times (\Gamma \cup \{\varepsilon\})\right) \times (Q \times \Gamma^*)$: transition relation

$q_0 \in Q$: start state $\qquad F \subseteq Q$: set of accepting states

How do we interpret $((q, a, C), (q', D_1 D_2 \cdots D_k)) \in \Delta$?

Whenever the machine is in state q reading letter $a \in \Sigma$,
and the symbol $C \in \Gamma$ is on the top of the stack,
it can
- pop C off the stack (if $C = \varepsilon$, no need to pop anything)
- push D_k, then $D_{k-1}, \ldots,$ then D_1 onto the stack.
- move to state q', and read the next input letter.

If a is ε, do the same thing, but without reading anything!

What all information does one need in order to fully specify the behaviour of a PDA?

→ Current state

→ Whatever string the PDA is going to read

→ Current stack contents

A Configuration of a PDA $M = (Q, \Sigma, \Gamma, \Delta, q_0, F)$ is $c \in Q \times \Sigma^* \times \Gamma^*$, which fixes these three parameters.

What is the start configuration on an input word ω?

$(q_0, \omega, \perp)$

What can the machine M do in one step from a configuration?

Suppose $((q, a, A), (q', s)) \in \Delta$. Then, we say that

$\omega \in \Sigma^*$ $\quad S \in \Gamma^*$

$$(q, a\omega, AS) \xrightarrow[M]{1} (q', \omega, sS),$$

for any $\omega \in \Sigma^*$, $S \in \Gamma^*$. If $a = \varepsilon$, $a\omega = \omega$.

Suppose $c, d \in Q \times \Sigma^* \times \Gamma^*$ are configurations of M. Then,

$c \xrightarrow[M]{0} d$ iff $c = d$

$c \xrightarrow[M]{n+1} d$ iff there is some c' s.t. $c \xrightarrow[M]{n} c'$ and $c' \xrightarrow[M]{1} d$.

$c \xrightarrow[M]{*} d$ iff there is some $n \geq 0$ s.t. $c \xrightarrow[M]{n} d$.

What strings does M accept? $\{a^n b^n aab \mid n \geq 0\}$

One can specify acceptance in one of two (equivalent) ways

→ By final state: M accepts ω by final state if
$(q_0, \omega, \perp) \xrightarrow[M]{*} (f, \varepsilon, s)$ for some $s \in \Gamma^*$ and some $f \in F$.

→ By empty stack: M accepts ω by empty stack if
$(q_0, \omega, \perp) \xrightarrow[M]{*} (q, \varepsilon, \perp)$ for some $q \in Q$.
(F is irrelevant here!)

* These criteria are actually equivalent!
Either kind of machine can simulate the other.

$L = \{a^n b^n \mid n \geq 0\}$ $\quad M = (\{q_0, q_1\}, \{a, b\}, \{\perp, C\}, \Delta, q_0, F)$

$$\Delta = \{((q_0, a, \varepsilon), (q_0, C)), ((q_0, b, C), (q_1, \varepsilon)), ((q_1, b, C), (q_1, \varepsilon))\}$$

What configurations does M go through on input word a^4b^4?

$$(q_0, a^4b^4, \perp) \xrightarrow[M]{1} (q_0, a^3b^4, C\perp) \xrightarrow[M]{1} (q_0, a^2b^4, CC\perp) \rightarrow (q_0, a^1b^4, CCC\perp) \rightarrow (q_0, b^4, CCCC\perp) \rightarrow (q_1, b^3, CCC\perp) \rightarrow (q_1, b^2, CC\perp) \rightarrow (q_1, b, C\perp) \rightarrow (q_1, \varepsilon, \perp)$$

When does M accept? By empty stack

M accepts ω iff $(q_0, \omega, \perp) \xrightarrow[M]{*} (q, \varepsilon, \perp)$ for some $q \in Q$