Context-Free Grammars

Recall: Regular languages had regexes as a representation

We saw that for some non-regular languages, we could write grammars to represent them.

A grammar is a 4-tuple of the form (NT, T, R, S)

- NT: Set of non-terminals
- T: Set of terminals
- R: Set of production rules
- S: $\in NT$ Start symbol

Each production rule in a context-free grammar has a non-terminal on the left, followed by $::=$ or $\longrightarrow$, and some sequence of terminals & non-terminals on the right

We gave a grammar for

$\mathcal{L} = \{ \omega \mid \omega$ contains as many 'a's as 'b's$\} \subseteq \{a, b\}^*$

as follows

$$S ::= \varepsilon \mid aSb \mid bSa \mid SS$$

$$G = (NT, T, R, S)$$

where $NT = \{S\}$

$T = \{a, b\}$

$R = \{ S ::= \varepsilon,\ S ::= aSb,\ S ::= bSa,\ S ::= SS \}$

The main use of CFGs is in parsing.

For example, I might want to recognize the language of all strings with balanced parentheses

But the same CFG from above does not work for this!

$$\mathcal{L}_{()} = \left\{ w \;\middle|\; \begin{array}{l} w \text{ contains an equal number of `(' and `)', and} \\ \text{every prefix } s \text{ of } w \text{ contains at least} \\ \text{as many `('s as `)'s} \end{array} \right\}$$

This is a very verbose description.

What is a CFG for $\mathcal{L}$?

$S ::= \varepsilon \mid (S) \mid SS$

$G = (\{S\}, \{(,)\}, \{S ::= \varepsilon, S ::= (S), S ::= SS\}, S)$

How do we prove that $\mathcal{L}(G) = \mathcal{L}_{()}$?

① Show that every string $w \in \mathcal{L}(G)$ is s.t. $w \in \mathcal{L}_{()}$.

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Possible cases for w:

$w = \varepsilon$: $\quad S ::= \varepsilon$ ✓

$\omega = (\omega'($: $\times$

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$\omega =)\omega')$: $\times$

$\omega = (\omega')$: Let ω_p be the leftmost prefix of (ω') s.t.

$$\#\text{'('}(\omega_p) = \#\text{')'}(\omega_p)$$

Let $\omega = \omega_p \cdot \omega_s$

(a) $\omega_p = \omega$: $S ::= (S)$ $\quad S ::= \omega'$

(b) $\omega_p \neq \omega$: $S ::= \omega_p$ $\quad S ::= \omega_s$ $\quad S ::= SS$